

The spectral shift function for planar obstacle scattering at low energy

I McGillivray
 School of Mathematics
 University of Bristol
 University Walk
 Bristol BS8 1TW
 United Kingdom
 e maiemg@bristol.ac.uk
 t + 44 (0)117 3311663
 f + 44 (0)117 9287999

Abstract

Let H signify the free non-negative Laplacian on \mathbb{R}^2 and H_Y the non-negative Dirichlet Laplacian on the complement Y of a nonpolar compact subset K in the plane. We derive the low-energy expansion for the Krein spectral shift function (scattering phase) for the obstacle scattering system $\{H_Y, H\}$ including detailed expressions for the first three coefficients. We use this to investigate the large time behaviour of the expected volume of the pinned Wiener sausage associated to K .

Key words: obstacle scattering, regularised heat-trace, Krein spectral shift function, scattering phase, pinned Wiener sausage, Brownian bridge

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1 Introduction

Given a nonpolar compact subset K in the plane, we consider the exterior domain Y complementary to K . Let \mathcal{H} stand for the Hilbert space $L^2(\mathbb{R}^2, m)$ where m signifies the Lebesgue measure. The (non-negative) Laplacian acting in \mathcal{H} will be denoted H while H_Y denotes the (non-negative) Dirichlet Laplacian acting in $L^2(Y, m)$. The operator J embeds $L^2(Y, m)$ into \mathcal{H} through extending by zero on Y . Let $\xi(\lambda)$ stand for the Krein spectral shift function (scattering phase) for the pair $\{H_Y, H\}$. The following trace formula then holds

$$\mathrm{Tr} [Jg(H_Y)J^* - g(H)] = \int_0^\infty g'(\lambda)\xi(\lambda) d\lambda \quad (1.1)$$

for any function $g : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$g(\lambda) = \begin{cases} f(e^{-\lambda}) & \text{for } \lambda > 0, \\ 0 & \text{for } \lambda \leq 0, \end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function with $f' \in W^{1,2}(\mathbb{R})$. In this paper, we study the asymptotic behaviour of $\xi(\lambda)$ for small λ . Our main result is this.

Theorem 1.1. *Let K be a nonpolar compact subset of \mathbb{R}^2 . Let $l \in \mathbb{N}$. There exist $\xi_0^k \in \mathbb{R}$ ($-\infty < k \leq -1$) such that*

$$\xi(\lambda) = \sum_{k=-l}^{-1} \xi_0^k (-\log \lambda)^k + o((-\log \lambda)^{-l})$$

as $\lambda \downarrow 0$. The first three coefficients are given by

$$(i) \quad \xi_0^{-1} = 1;$$

- (ii) $\xi_0^{-2} = C(K) - \log 4 + 2\gamma;$
- (iii) $\xi_0^{-3} = (C(K) - \log 4 + 2\gamma)^2 - \frac{\pi^2}{3}.$

The quantity $C(K)$ is related to the Robin constant $R(K)$ of K via the relation $C(K) = -4\pi R(K)$; γ stands for Euler's constant. We remark that the leading order expansion for $\xi(\lambda)$ has been derived in [15]. The counterpart to this result in higher dimensions may be found in [18], [19].

The study is motivated in part by its relevance to a problem in probability theory. The pinned Wiener sausage $S(t, \omega)$ refers to the random set swept out by a compact set K in the plane as it is transported along a Brownian loop $\omega : [0, t] \rightarrow \mathbb{R}^2$ by rigid motion. In detail,

$$S(t, \omega) := \bigcup_{0 \leq s \leq t} (\omega(s) + K) \subseteq \mathbb{R}^2;$$

its area is denoted $|S(t, \omega)|$. Introduce the expected area via

$$\gamma(t) := \mathbb{E}_{0,0}^{0,t} |S(t, \omega)|$$

where $\mathbb{P}_{0,0}^{0,t}$ signifies the Brownian bridge measure on loop space associated to the Laplacian Δ . We derive an asymptotic expansion for the above quantity in the large time régime.

Theorem 1.2. *Let K be a nonpolar compact subset of \mathbb{R}^2 . Let $l \in \mathbb{N}$. Then there exist $\gamma_0^k \in \mathbb{R}$ ($\mathbb{Z} \ni k \leq -1$) such that*

$$\gamma(t) = \sum_{k=-l}^{-1} \gamma_0^k t (\log t)^k + o(t (\log t)^{-l})$$

as $t \rightarrow \infty$. The first three coefficients are given by

- (i) $\gamma_0^{-1} = 4\pi;$
- (ii) $\gamma_0^{-2} = 4\pi \{ C(K) + \gamma - \log 4 \};$
- (iii) $\gamma_0^{-3} = 4\pi \left\{ (C(K) + \gamma - \log 4)^2 - \frac{\pi^2}{6} \right\}.$

This result was conjectured in [5]. This latter work derives the first order asymptotic expansion of $\gamma(t)$ for an arbitrary nonpolar compact K . It also obtains the third order asymptotic series as above for the particular case in which $K = K_a$ is a closed disc with radius $a > 0$.

It is interesting to compare the behaviour of $\gamma(t)$ with the related functional $\beta(t) := \mathbb{E}_0 |S(t, \omega)|$. Here $\omega : [0, \infty) \rightarrow \mathbb{R}^2$ is a Brownian path in \mathbb{R}^2 and \mathbb{P}_0 stands for the Wiener measure on path space associated to Δ . In the large time régime

$$\beta(t) = \sum_{k=-l}^{-1} \beta_0^k t (\log t)^k + o(t (\log t)^{-l})$$

with explicit expressions for the first three coefficients according to [13]. This last expansion extends the work of [23] which detailed the second order expansion. The series for $\gamma(t)$ and $\beta(t)$ agree to leading order. For the lower order terms we have that

$$\beta_0^{-2} = 4\pi \{ C(K) + 1 + \gamma - \log 4 \} \quad \text{while} \quad \beta_0^{-3} = 4\pi \left\{ (C(K) + 1 + \gamma - \log 4)^2 - \frac{\pi^2}{6} \right\}.$$

The analogous problem for γ in higher dimensions has been treated in [18], [19]. This problem originated in the calculation of the specific heat of a quantum system of obstacles K at low temperature [28].

In Section 2 we introduce the trace formula (1.1) and relate the Krein spectral shift function $\xi(\lambda)$ to the scattering matrix $S(\lambda)$ for the system $\{H_Y, H\}$ via the Birman-Kreĭn formula. We show that this relation holds in particular on an interval of the form $(0, \delta)$, including the case when K does not have a connected complement. In Section 3 we derive a number of prerequisite results in logarithmic potential theory.

In order to construct the scattering matrix $S(\lambda)$ it is necessary to invert the operator

$$I + R^{(-1)}(\mu - \imath 0)V \quad (1.2)$$

in $B(\mathcal{H}_{-s})$ for $\lambda > 0$ in a neighbourhood of $\lambda = 0$; here, μ relates to λ via $\mu = (\lambda + 1)^{-1}$. To explain terminology briefly,

$$V = JR_Y(-1)J^* - R(-1)$$

denotes the difference between the Dirichlet and free resolvents; while $R^{(-1)}(\cdot)$ signifies the resolvent of $R(-1)$. Also, \mathcal{H}_{-s} refers to a weighted Hilbert space. The operator in (1.2) explodes on the complement of a hyperspace in \mathcal{H}_{-s} . This complication is absent in higher dimensions; it presents the salient technical difficulty of the paper. This is tackled in Sections 4 and 6.

Section 6 continues with a small energy expansion of the scattering matrix $S(\lambda)$ in a double-series akin to expansions obtained in [4], [14]. A lattice-point counting lemma in Section 5 plays a role in establishing summability of the double-series. Section 6 culminates in the proof of the expansion given in Theorem 1.1. The detailed derivation of the coefficients is left to Section 7. The application Theorem 1.2 is proved in Section 8. The Appendix includes the proofs of several results from [18].

2 The trace formula

The free Laplacian. Let \mathcal{H} stand for the complex Hilbert space $L^2(\mathbb{R}^2, m)$ based on Lebesgue measure m with inner product (\cdot, \cdot) linear in the first factor. We refer to the non-negative Laplacian $-\Delta$ in \mathcal{H} by H . Its resolvent $R(\zeta) := (H - \zeta)^{-1}$ ($\zeta \in \mathbb{C} \setminus [0, \infty)$) has convolution kernel $k(x; \zeta)$ given by

$$k(x; \zeta) := \frac{\imath}{4} H_0^{(1)}(\zeta^{1/2}|x|) \quad (2.1)$$

where $H_0^{(1)}$ is the first Hankel function of order 0. The condition $\Im \zeta^{1/2} > 0$ specifies the branch of $\zeta^{1/2}$. For the sake of completeness, we recall that

$$\begin{aligned} H_0^{(1)}(z) &= 1 + \frac{2}{\pi} \gamma \imath - \left\{ 1 + \frac{2}{\pi} \imath(\gamma - 1) \right\} \frac{z^2/4}{(1!)^2} + \left\{ 1 + \frac{2}{\pi} \imath(\gamma - 1 - \frac{1}{2}) \right\} \frac{(z^2/4)^2}{(2!)^2} + \dots \\ &+ \frac{2}{\pi} \gamma \imath \text{Log}(z/2) \left\{ 1 - \frac{z^2/4}{(1!)^2} + \frac{(z^2/4)^2}{(2!)^2} - \dots \right\} \quad (z \in \mathbb{C} \setminus [0, \infty)) \end{aligned} \quad (2.2)$$

as in [1] 9.1.3, 9.1.12, 9.1.13. The logarithm Log refers to the principal branch of the logarithm.

Let us introduce constants

$$a_j := \begin{cases} (1/2\pi) (\log 2 - \gamma) + \imath/4, & j = 0, \\ \left\{ \frac{1}{2\pi} \left(\log 2 - \gamma - \sum_{k=1}^j \frac{1}{k} \right) + \frac{\imath}{4} \right\} \frac{(-1)^j}{4^j (j!)^2}, & j \geq 1; \end{cases} \quad (2.3)$$

$$b_j := \begin{cases} -1/2\pi, & j = 0, \\ \frac{(-1)^{j+1}}{2\pi} \frac{1}{4^j (j!)^2}, & j \geq 1; \end{cases} \quad (2.4)$$

$$c_j := \frac{1}{4\pi} \frac{(-1)^j}{4^j (j!)^2}, \quad j \geq 0. \quad (2.5)$$

Put

$$k_j^0(x) = \{a_j + b_j \log |x|\} |x|^{2j} \text{ and } k_j^1(x) = c_j |x|^{2j} \quad (x \in \mathbb{R}^2 \setminus \{0\}).$$

Then

$$k(x; \zeta) = \sum_{j=0}^{\infty} \sum_{\varepsilon=0}^1 \zeta^j \eta^\varepsilon k_j^\varepsilon(x) \quad (x \in \mathbb{R}^2 \setminus \{0\}) \quad (2.6)$$

with $\eta := -2 \operatorname{Log} \zeta^{1/2}$.

Define $\langle x \rangle := (1 + |x|^2)^{1/2}$ for $x \in \mathbb{R}^2$. The weighted L^2 -space \mathcal{H}_s ($s \in \mathbb{R}$) is defined by $\mathcal{H}_s := \{u : \langle \cdot \rangle^s u \in \mathcal{H}\}$. Considered as Banach spaces, the dual space of \mathcal{H}_s is \mathcal{H}_{-s} . We write $\langle \cdot, \cdot \rangle$ for the corresponding duality pairing.

According to [3] Theorem 4.1,

$$R(\lambda \pm i0) := \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon)$$

exists in $B(\mathcal{H}_s, \mathcal{H}_{-s})$ for any $s > 1/2$ and $\lambda > 0$, with convergence in the uniform operator topology. Further,

Theorem 2.1. *Let $l \in \mathbb{N}_0$ and $s > 2l + 1$. Then for $\zeta \in \mathbb{C} \setminus [0, \infty)$,*

$$\left\| R(\zeta) - \sum_{j=0}^l \sum_{\varepsilon=0}^1 \zeta^j \eta^\varepsilon K_j^\varepsilon \right\|_{B(\mathcal{H}_s, \mathcal{H}_{-s})} = o(|\zeta|^l)$$

as $\zeta \rightarrow 0$ where K_j^ε is the operator with convolution kernel k_j^ε .

This is proved in the Appendix (see also Proposition 3.7 in [18]).

The modified resolvent $R^{(-1)}(\zeta)$ is the resolvent of $R(-1)$. It relates to the resolvent of H via

$$R^{(-1)}((1 + \zeta)^{-1}) = -(1 + \zeta) (I + (1 + \zeta)R(\zeta)), \quad \zeta \in \mathbb{C} \setminus [0, \infty).$$

The auxiliary Hilbert space $L^2(S^1, \sigma)$ is denoted by \mathfrak{h} . Let $U : \mathcal{H} \rightarrow L^2((0, \infty); \mathfrak{h})$ be the spectral representation of H . Then for any $u \in \mathcal{H}_s$ and $\lambda > 0$,

$$U(\lambda)u(\omega) = (1/\sqrt{2})\mathcal{F}u(\lambda^{1/2}\omega) \quad (\omega \in S^1)$$

provided that $s > 1$. Here, \mathcal{F} stands for the Fourier transform

$$\mathcal{F}u(\xi) = \hat{u}(\xi) := (2\pi)^{-1} \int_{\mathbb{R}^2} e^{-i\xi \cdot x} u(x) m(dx).$$

The following lemma is proved in the Appendix (see also [18] Lemma 3.9).

Lemma 2.1. *Fix $l \in \mathbb{N}_0$ and $s > l + 1$. Then*

$$\left\| U(\lambda) - \sum_{j=0}^l (i\lambda^{1/2})^j U_j \right\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathfrak{h})} = o(\lambda^{l/2})$$

as $\lambda \downarrow 0$. The operator U_j has kernel

$$u_j(\omega, x) = \frac{1}{\sqrt{2}} (2\pi)^{-1} \frac{(-1)^j}{j!} (\omega \cdot x)^j. \quad (2.7)$$

Incidentally, the notation $\mathfrak{S}_2(\mathcal{H}_s, \mathfrak{h})$ refers to the collection of operators from \mathcal{H}_s to \mathfrak{h} of Hilbert-Schmidt type.

Let $U^{(-1)} : \mathcal{H} \rightarrow L^2((0, 1); \mathfrak{h})$ be the spectral representation of $R(-1)$. Then U and $U^{(-1)}$ are related via

$$U^{(-1)}(\mu) = (\lambda + 1)U(\lambda). \quad (2.8)$$

We use the notation

$$\mu = (\lambda + 1)^{-1}$$

and this is used routinely in the sequel. Let $l \in \mathbb{N}_0$ and $s > l + 1$. From Lemma 2.1 we derive

$$U^{(-1)}(\mu) = \sum_{j=0}^{2l} (i\lambda^{1/2})^j U_j^{(-1)} + o(\lambda^l) \quad (2.9)$$

in $\mathfrak{S}_2(\mathcal{H}_s, \mathfrak{h})$ as $\lambda \downarrow 0$ where

$$\begin{aligned} U_0^{(-1)} &= U_0, \\ U_1^{(-1)} &= U_1, \\ U_j^{(-1)} &= U_j - U_{j-2} \text{ for } j \geq 2. \end{aligned} \quad (2.10)$$

The spectral shift function. Let K be a nonpolar compact subset of \mathbb{R}^2 . Its complement will be denoted by Y . Let H_Y refer to the non-negative Dirichlet Laplacian on $L^2(Y, m)$. The semigroup difference

$$J e^{-H_Y} J^* - e^{-H} \in \mathfrak{S}_1(\mathcal{H}) \quad (2.11)$$

is trace class [24]. Let $\xi(\lambda, e^{-H_Y}, e^{-H})$ be the spectral shift function for the pair $\{e^{-H_Y}, e^{-H}\}$ ([31] Theorem 8.2.1). Define

$$\xi(\lambda) = \xi(\lambda, H_Y, H) := \begin{cases} -\xi(e^{-\lambda}, e^{-H_Y}, e^{-H}), & \lambda \geq 0, \\ 0, & \lambda < 0. \end{cases} \quad (2.12)$$

By [31] Theorem 8.2.1,

$$\xi \in L^1(\mathbb{R}; e^{-|\lambda|} d\lambda). \quad (2.13)$$

By [31] Theorem 8.3.3 and the paragraph following it, we may write

$$\text{Tr} [Jg(H_Y)J^* - g(H)] = \int_0^\infty g'(\lambda) \xi(\lambda) d\lambda \quad (2.14)$$

for any function $g : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$g(\lambda) = \begin{cases} f(e^{-\lambda}) & \text{for } \lambda > 0, \\ 0 & \text{for } \lambda \leq 0, \end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuously differentiable function with $f' \in W^{1,2}(\mathbb{R})$. In particular, given $t > 3/2$ we can find a continuously differentiable function f such that $f(\lambda) = \lambda^t$ for $0 \leq \lambda \leq 1$ and $f' \in W^{1,2}(\mathbb{R})$. We then have

$$\text{Tr} [J e^{-tH_Y} J^* - e^{-tH}] = - \int_0^\infty t e^{-t\lambda} \xi(\lambda) d\lambda \quad (2.15)$$

for $t > 3/2$.

Let Y_e stand for the unbounded connected component of Y and set $Y_b := Y \setminus Y_e$. In case $Y_b \neq \emptyset$, we differentiate between H_e resp. H_b , the non-negative Dirichlet Laplacians on Y_e resp. Y_b . The spectrum $\sigma(H_b)$ of H_b is discrete. Let $\xi_e(\lambda) = \xi(\lambda, H_{Y_e}, H)$ be the spectral shift function for the pair $\{H_{Y_e}, H\}$. Denote by

$$N_b(\lambda) := \sum_{\sigma(H_b) \ni \nu < \lambda} m(\nu)$$

the spectral counting function for H_b ; here, $m(\nu)$ stands for the geometric multiplicity of $\nu \in \sigma(H_b)$.

Lemma 2.2. *It holds that*

$$(i) \quad \xi(\lambda) = \xi_e(\lambda) + N_b(\lambda) \text{ for a.e. } \lambda > 0;$$

$$(ii) \quad \xi \text{ admits an a.e.-version that is real analytic on } (0, \infty) \setminus \sigma(H_b).$$

Finally, with ξ denoting this version,

$$(iii) \quad \xi(\lambda) \rightarrow 0 \text{ as } \lambda \downarrow 0.$$

Proof. First note that both ξ and ξ_e satisfy (2.13). For $t > 3/2$,

$$\int_0^\infty e^{-t\lambda} \xi(\lambda) d\lambda = t^{-1} \operatorname{Tr} [e^{-tH} - J_e e^{-tH_{Y_e}} J_e^*] - t^{-1} \operatorname{Tr} [e^{-tH_{Y_b}}] = \int_0^\infty e^{-t\lambda} \{ \xi_e(\lambda) + N_b(\lambda) \} d\lambda$$

where the Weyl asymptotics of $N_b(\cdot)$ ensure the absolute integrability of the second integrand. Item (i) follows by the inversion formula for the Laplace-Lebesgue integral ([30] Theorem VII.6a). Parts (ii) and (iii) follow from [16] Lemmas 3.2 and 3.4 and (i). \square

The scattering matrix. In virtue of (2.11) the scattering operator $S(e^{-H_Y}, e^{-H}, J)$ for the pair $\{e^{-H_Y}, e^{-H}\}$ exists and is unitary on \mathcal{H} by [31] Theorem 6.2.1 and Corollary 2.4.2. By the invariance principle ([31] Theorem 6.2.5), the scattering operator $S(R_Y(-1), R(-1), J)$ exists and $S(R_Y(-1), R(-1), J) = S(e^{-H_Y}, e^{-H}, J)$. As the scattering operators commute with the corresponding spectral projectors ([31] Theorem 2.1.4 and 1.5.1) we have the representation

$$S(\lambda, e^{-H_Y}, e^{-H}, J) = S(\varphi(\lambda), R_Y(-1), R(-1), J) \quad \text{a.e. } \lambda > 0$$

where $\varphi : (0, 1) \rightarrow (0, 1); \lambda \mapsto (-\log \lambda + 1)^{-1}$. By the Birman-Kreĭn formula ([31] Theorem 8.4.1),

$$\begin{aligned} e^{2\pi i \xi(\lambda)} &= \operatorname{Det}(S(e^{-\lambda}, e^{-H_Y}, e^{-H}, J)) \\ &= \operatorname{Det}(S(\mu, R_Y(-1), R(-1), J)) \quad \text{a.e. } \lambda > 0. \end{aligned} \tag{2.16}$$

We now derive a representation formula for $S(\mu, R_Y(-1), R(-1), J)$.

Set

$$V := J R_Y(-1) J^* - R(-1).$$

Then

Theorem 2.2. *For each $s > 0$, V admits a bounded extension from \mathcal{H}_{-s} to \mathcal{H}_s and $V = V^* \in \mathfrak{S}_\infty(\mathcal{H}_{-s}, \mathcal{H}_s)$ is compact.*

This result is proved in the Appendix (see also [18] Theorem 2.1). Given $s > 1/2$ and $\mu \in (0, 1)$, define

$$\mathcal{H}_s^\mu := \{ f = (R(-1) - \mu) u : u \in \mathcal{H}_s \}.$$

Lemma 2.3. *Let $s > 1/2$ and $\mu \in (0, 1)$. We have*

- (i) $R(-1) \in B(\mathcal{H}_s, \mathcal{H}_s)$;
- (ii) \mathcal{H}_s^μ is a proper subspace of \mathcal{H}_s ;
- (iii) the identity

$$R^{(-1)}(\mu \pm \imath 0) (R(-1) - \mu) = I$$

holds on \mathcal{H}_s .

Proof. (i) From the identity $\langle x \rangle^s \leq 2^s \{ \langle y \rangle^s + \langle x - y \rangle^s \}$ obtain

$$\langle x \rangle^s k(x - y; -1) \langle y \rangle^{-s} \leq k(x - y; -1) + \langle x - y \rangle^s k(x - y; -1) \quad (x \neq y).$$

The latter kernel is integrable by [25] 3.6, so defines a bounded convolution operator on \mathcal{H} by Young's inequality [8].

(ii) Let us introduce the Sobolev space $W_s := \{u : \widehat{u} \in \mathcal{H}_s\}$. Let $\tau : W_s \rightarrow L^2(S_\lambda^1, \sigma)$ stand for the restriction mapping ([22] Theorem IX.39). Then $\tau \widehat{f} = 0$ for any $f \in \mathcal{H}_s^\mu$. The function $f = e^{-|\cdot|^2/2} \in \mathcal{H}_s$ does not satisfy this property as $\widehat{f} = f$ ([17] Theorem 5.2).

(iii) Let $u \in \mathcal{H}_s$ and $f := (R(-1) - \mu)u$. Then

$$\begin{aligned} \left\| u - R^{(-1)}(\mu - \imath 0)f \right\|_{\mathcal{H}_{-s}} &= \lim_{\varepsilon \downarrow 0} \left\| u - R^{(-1)}(\mu - \imath \varepsilon)f \right\|_{\mathcal{H}_{-s}} \\ &= \lim_{\varepsilon \downarrow 0} \left\| u - R^{(-1)}(\mu - \imath \varepsilon) [R(-1) - (\mu - \imath \varepsilon) - \imath \varepsilon] u \right\|_{\mathcal{H}_{-s}} \\ &= \lim_{\varepsilon \downarrow 0} \varepsilon \left\| R^{(-1)}(\mu - \imath \varepsilon)u \right\|_{\mathcal{H}_{-s}} \\ &= 0 \end{aligned}$$

and similar with the opposite sign. □

Theorem 2.3. *Let $s > 1/2$.*

- (i) *Assume that $Y_b = \emptyset$. For any $\mu \in (0, 1)$, the compact operator $VR^{(-1)}(\mu \pm \imath 0)$ acting in $B(\mathcal{H}_s)$ does not have eigenvalue -1 .*
- (ii) *Assume that $Y_b \neq \emptyset$ and $\lambda \notin \sigma(H_b)$. Then $VR^{(-1)}(\mu \pm \imath 0)$ acting in $B(\mathcal{H}_s)$ does not have eigenvalue -1 .*
- (iii) *Assume that $Y_b \neq \emptyset$ and $\lambda \in \sigma(H_b)$. Then $VR^{(-1)}(\mu \pm \imath 0)$ acting in $B(\mathcal{H}_s)$ has eigenvalue -1 .*

Proof. (i) Suppose that $VR^{(-1)}(\mu - \imath 0)f = -f$ for some $f \in \mathcal{H}_s$. Set $u := R^{(-1)}(\mu - \imath 0)f$. Argue as in [7] Lemma 4.4 to conclude that $u \in \mathcal{H}$ and that $R_Y(-1)u = \mu u$. Put $f := \Delta u + \lambda u \in \mathcal{D}'(\mathbb{R}^2)$ with $\lambda := -1 + 1/\mu$. Then $f = 0$ on Y because u is a weak solution of $\Delta u + \lambda u = 0$ there. By elliptic regularity [25] Proposition 3.9.1, u is smooth on Y . Adapting the argument in [26] Lemma 1.2 to the $d = 2$ case, conclude that u vanishes on the complement of some ball $B(0, r)$. The unique continuation property ([11] Theorem 5.1, for example) ensures that u vanishes throughout Y . The proof of (ii) is similar.

(iii) Let $\varphi \in L^2(Y_b)$ be an eigenfunction of H_b corresponding to λ . Let $u \in \mathcal{H}_s$ be the extension of φ by 0. Then $f := [R(-1) - \mu]u \in \mathcal{H}_s$ by Lemma 2.3 (i). Also, $R^{(-1)}(\mu - \imath 0)f = u$ by Lemma 2.3 (iii). In an obvious notation,

$$Vu = [R_{Y_b}(-1) \oplus R_{Y_c}(-1) - R(-1)]u = -[R(-1) - \mu]u = -f;$$

that is, $VR^{(-1)}(\mu - \imath 0)f = -f$. □

For $\lambda \in (0, \infty) \setminus \sigma(H_b)$,

$$\exists (I + VR^{(-1)}(\mu + i0))^{-1} \in B(\mathcal{H}_s)$$

by the Fredholm alternative. As in [31] Theorem 5.7.1' (with $\mathfrak{G} = \mathcal{H}_s$ for $s > 1/2$ and $G : \mathcal{H} \rightarrow \mathfrak{G}; f \mapsto \langle \cdot \rangle^{-2s} f$) the scattering matrix for $\{R_Y(-1), R(-1)\}$ can be represented

$$S(\mu, R_Y(-1), R(-1), J) = I - 2\pi i U^{(-1)}(\mu) (I + VR^{(-1)}(\mu + i0))^{-1} V U^{(-1)}(\mu)^*, \quad \text{a.e. } \lambda \in (0, \infty) \setminus \sigma(H_b).$$

Let $S(\cdot)$ stand for the (adjoint) scattering matrix for $\{H, H_Y\}$,

$$S(\lambda) = I + 2\pi i U^{(-1)}(\mu) V (I + R^{(-1)}(\mu - i0) V)^{-1} U^{(-1)}(\mu)^* \in B(\mathfrak{h})$$

with $\lambda \in (0, \infty) \setminus \sigma(H_b)$. The t -matrix is characterised by the relation $S(\lambda) = I + T(\lambda)$. From (2.17),

$$e^{-2\pi i \xi(\lambda)} = \text{Det } S(\lambda) \quad \text{a.e. } \lambda > 0. \quad (2.17)$$

3 Some logarithmic potential theory

For brevity, we use the notation g^λ to stand for the resolvent operator $R(-\lambda)$ with $\lambda > 0$; $g^\lambda(\cdot)$ stands for the corresponding convolution kernel. If $\lambda = 0$ the notation g is sometimes used. For $z \in \mathbb{C} \setminus (-\infty, 0]$ define

$$b(z) := H_0^{(1)}(z) - 1 - \frac{2i}{\pi} \{ \text{Log}(z/2) + \gamma \}. \quad (3.1)$$

Given $0 < \delta < 1$ there exists a finite constant c such that

$$|b(z)| \leq C |z|^2 (-\log |z|) \text{ for } |z| \leq \delta. \quad (3.2)$$

The logarithmic potential kernel is defined by

$$k(x) := (1/2\pi) \log(1/|x|) \quad (x \in \mathbb{R}^2 \setminus \{0\}). \quad (3.3)$$

From (2.1), the kernel $g^\lambda(\cdot)$ may be decomposed in terms of (3.3) and (3.1) as

$$g^\lambda(x) = a_0 - i\pi c_0 - c_0 \log \lambda + k(x) + r(\lambda^{1/2}|x|) \quad (x \in \mathbb{R}^2 \setminus \{0\}) \quad (3.4)$$

with a_0 and c_0 as in (2.3) and (2.5), and

$$r(x) := (i/4)b(ix).$$

Fix a unit vector u in \mathbb{R}^2 . The regularised resolvent kernel $k^\lambda(\cdot)$ is given by

$$k^\lambda(x) := g^\lambda(x) - g^\lambda(u) \quad (x \in \mathbb{R}^2 \setminus \{0\}). \quad (3.5)$$

From (3.4),

$$g^\lambda(u) = a_0 - i\pi c_0 - c_0 \log \lambda + r(\lambda^{1/2}),$$

and hence

$$k^\lambda(x) - k(x) = r(\lambda^{1/2}|x|) - r(\lambda^{1/2}) \quad (x \in \mathbb{R}^2 \setminus \{0\}). \quad (3.6)$$

The operators with convolution kernels $k^\lambda(\cdot)$ resp. $k(\cdot)$ will be denoted by k^λ resp. k . We use the notation r^λ to refer to the operator with convolution kernel $r(\lambda^{1/2}|x|)$.

Corollary 3.1. *Let $s > 1$. Then $k^\lambda \rightarrow k$ in $B(\mathcal{H}_s, \mathcal{H}_{-s})$ as $\lambda \downarrow 0$.*

Proof. We may write

$$k^\lambda - k = g^\lambda - \eta K_0^1 - K_0^0 - r(\lambda^{1/2}) \langle \cdot, 1 \rangle 1$$

with η as before given by $\eta = -2 \operatorname{Log}(-\lambda)^{1/2} = -\log \lambda - i\pi$. Now apply Theorem 2.1 and (3.2). \square

Theorem 3.1. *Let $s > 1$. Then*

- (i) $k : \mathcal{H}_s \rightarrow C(\mathbb{R}^2)$;
- (ii) $r^\lambda : \mathcal{H}_s \rightarrow C(\mathbb{R}^2)$ for each $\lambda > 0$;
- (iii) $g^\lambda(u) r^\lambda f \rightarrow 0$ locally uniformly on \mathbb{R}^2 as $\lambda \downarrow 0$ for each $f \in \mathcal{H}_s$.

Proof. (i) Define

$$k_1 f(x) := -(1/2\pi) \int_{B(x,1)} \log |x-y| f(y) m(dy),$$

and likewise for k_2 but with $B(x,1)$ replaced by its complement $B(x,1)^c$. Note that

$$\left| \int_{B(x,r)} f(y) m(dy) \right| \leq \sqrt{\pi} \|f\|_{\mathcal{H}_s} r$$

for $f \in \mathcal{H}_s$. As in [2] Lemma 3.1.1 (b),

$$k_1 f(x) = (1/2\pi) \int_0^1 \int_{B(x,r)} f(y) m(dy) \frac{dr}{r} + (1/2\pi) \int_{B(x,1)} f(y) m(dy).$$

A dominated convergence argument shows that $k_1 f$ is continuous on \mathbb{R}^2 . As for k_2 , we have

$$\chi_{B(x,1)^c}(y) \log |x-y| \leq \tau + |y| \quad (y \in \mathbb{R}^2)$$

for all $x \in B(0, \tau)$ ($\tau > 0$). Continuity of $k_2 f$ follows by another appeal to dominated convergence.

(ii) Write

$$r^\lambda = g^\lambda - k + \left\{ r(\lambda^{1/2}) - g^\lambda(u) \right\} \langle \cdot, 1 \rangle 1.$$

Now $\mathcal{H}_s \subseteq L^p(\mathbb{R}^2)$ for each $p > 1$. By the Sobolev embedding [2] Theorem 1.2.4, $g^\lambda : \mathcal{H}_s \rightarrow C(\mathbb{R}^2)$. This and (i) establish the claim.

(iii) Pick $0 < \alpha < 1/3$. For each $x \in \mathbb{R}^2$ introduce sets

$$\begin{aligned} A_1^\lambda &:= \{y \in \mathbb{R}^2 : |y-x| \leq 1\}, \\ A_2^\lambda &:= \{y \in \mathbb{R}^2 : 1 < |y-x| \leq \lambda^{-\alpha}\}, \\ A_3^\lambda &:= \{y \in \mathbb{R}^2 : \lambda^{-\alpha} < |y-x| \leq \delta \lambda^{-1/2}\}, \\ A_4^\lambda &:= \{y \in \mathbb{R}^2 : |y-x| > \delta \lambda^{-1/2}\}, \end{aligned}$$

for λ sufficiently small (where the x -dependence has been suppressed for the sake of legibility). Define

$$r_j^\lambda f(x) := \int_{A_j^\lambda} r(\lambda^{1/2}|x-y|) f(y) m(dy) \quad (x \in \mathbb{R}^2) \quad (j = 1, 2, 3, 4).$$

Fix $\tau > 0$. For $0 < \lambda < \delta^2$,

$$|r_1^\lambda f(x)| \leq (C/4) \lambda \left\{ \int_{|x-y| \leq 1} \left(\log \lambda^{1/2} |x-y| \right)^2 m(dy) \right\}^{1/2} \|f\|_{\mathcal{H}_s}$$

by (3.2). In particular, $g^\lambda(u) r_1^\lambda f \rightarrow 0$ uniformly on $B(0, \tau)$ as $\lambda \downarrow 0$.

Again from (3.2), for $\lambda > 0$ small,

$$|r_2^\lambda f(x)| \leq (C/4) \lambda \left\{ \int_{1 < |x-y| \leq \lambda^{-\alpha}} |x-y|^4 \left[(1/2)(\log \lambda)^2 + 2(\log |x-y|)^2 \right] m(dy) \right\}^{1/2} \|f\|_{\mathcal{H}_s}.$$

Choose $0 < \eta < 1/\alpha - 3$. An estimate of the form $|\log |x-y|| \leq c_\eta |x-y|^\eta$ holds on A_2^λ . Also,

$$\left\{ \int_{1 < |x-y| \leq \lambda^{-\alpha}} |x-y|^{4+2\eta} m(dy) \right\}^{1/2} \leq \left\{ \frac{\pi}{3+\eta} \right\}^{1/2} \lambda^{-\alpha(3+\eta)}.$$

So $g^\lambda(u) r_2^\lambda f \rightarrow 0$ uniformly on $B(0, \tau)$ as $\lambda \downarrow 0$.

The kernel $r(\lambda^{1/2}|x-y|)$ is bounded by a constant c' (say) on A_3^λ . Thus, for any $x \in B(0, \tau)$,

$$\begin{aligned} |r_3^\lambda f(x)| &\leq c' \int_{\lambda^{-\alpha} < |x-y| \leq \delta \lambda^{-1/2}} |f(y)| m(dy) \\ &\leq c' \int_{\lambda^{-\alpha} < |x-y|} |f(y)| m(dy) \\ &\leq c' \int_{B(0, \lambda^{-\alpha} - \tau)^c} |f(y)| m(dy) \\ &\leq c' \|f\|_{\mathcal{H}_s} \left\{ \int_{B(0, \lambda^{-\alpha} - \tau)^c} \langle y \rangle^{-2s} m(dy) \right\}^{1/2} \end{aligned}$$

provided λ is sufficiently small. The weight function $\langle \cdot \rangle^{-2s}$ is integrable because $s > 1$. Thus $g^\lambda(u) r_3^\lambda f \rightarrow 0$ uniformly on $B(0, \tau)$ as $\lambda \downarrow 0$.

By (3.4),

$$r_4^\lambda f(x) = \int_{|x-y| > \delta \lambda^{-1/2}} \left\{ g^\lambda(x-y) - a_0 + i\pi c_0 + c_0 \log \lambda - k(x-y) \right\} f(y) m(dy).$$

An estimate of the form (9.2) gives

$$\begin{aligned} \left| \int_{|x-y| > \delta \lambda^{-1/2}} g^\lambda(x-y) f(y) m(dy) \right| &\leq (c/4) \lambda^{-1/4} \int_{|x-y| > \delta \lambda^{-1/2}} |x-y|^{-1/2} |f(y)| m(dy) \\ &\leq (c/4) \delta^{-1/2} \lambda^{1/2} \int_{|x-y| > \delta \lambda^{-1/2}} |f(y)| m(dy). \end{aligned}$$

Uniform convergence can be derived in a way similar to the treatment of r_3^λ .

Choose $\eta > 0$ such that $s - \eta > 1$. For $x \in B(0, \tau)$ and λ small,

$$\begin{aligned} \left| \int_{|x-y| > \delta \lambda^{-1/2}} k(x-y) f(y) m(dy) \right| &\leq c_\eta \int_{|x-y| > \delta \lambda^{-1/2}} |x-y|^\eta |f(y)| m(dy) \\ &\leq c'_\eta \int_{|x-y| > \delta \lambda^{-1/2}} \langle y \rangle^\eta |f(y)| m(dy) \\ &\leq c'_\eta \int_{B(0, \delta \lambda^{-1/2} - \tau)^c} \langle y \rangle^\eta |f(y)| m(dy) \\ &\leq c'_\eta \|f\|_{\mathcal{H}_s} \left\{ \int_{B(0, \delta \lambda^{-1/2} - \tau)^c} \langle y \rangle^{-2(s-\eta)} m(dy) \right\}^{1/2} \end{aligned}$$

for appropriate constants c_η, c'_η . The remaining terms in r_4^λ can be dealt with using similar analysis. Consequently, $g^\lambda(u) r_4^\lambda f \rightarrow 0$ uniformly on $B(0, \tau)$ as $\lambda \downarrow 0$. \square

Let $M = (\Omega, \mathcal{M}, X_t, \mathbb{P}_x)$ be Brownian motion on \mathbb{R}^2 with transition function $p(t, \cdot)$ given by

$$p(t, x) = (4\pi t)^{-1} e^{-|x|^2/4t} \quad (t > 0)$$

(see [20] for example). Put $\sigma_K := \inf\{t > 0 : X_t \in K\}$, the first hitting time of K . The hitting operators h_K^λ are defined by

$$h_K^\lambda f := \mathbb{E} [e^{-\lambda \sigma_K} f(X_{\sigma_K}) : \sigma_K < \infty]$$

for measurable $f \geq 0$. If $\lambda = 0$, we write h_K instead of h_K^0 . The λ -potential of K is the function

$$p_K^\lambda := h_K^\lambda 1.$$

In case $\lambda = 0$, $p_K := p_K^0 \equiv 1$ by recurrence ([20] Proposition 2.9, for example). Set

$$W_K^\lambda := g^\lambda(u) \{1 - p_K^\lambda\}. \quad (3.7)$$

In case K is nonpolar, the limit

$$W_K := \lim_{\lambda \downarrow 0} W_K^\lambda \quad (3.8)$$

exists finitely according to [20] Theorem 3.4.2. The equilibrium measure μ_K is the unique probability measure concentrated on K^r whose potential $k\mu_K$ is constant on K^r . Here, K^r denotes the set of regular points for K ([20] 2.3). This constant value is the Robin constant $R(K)$ of K . We use the notation

$$C(K) := -4\pi R(K).$$

From [20] Theorem 3.4.12,

$$k\mu_K = R(K) - W_K. \quad (3.9)$$

In particular, $W_K \in \mathcal{H}_{-s}$ for any $s > 1$. The 1-capacity of K is denoted $C_1(K)$ (see [10]). It holds that

$$C_1(K) = \langle 1, p_K^1 \rangle. \quad (3.10)$$

We use the notation g_K^λ ($\lambda > 0$) to refer to the λ -potential operator with kernel

$$g_K^\lambda(x, y) := \int_0^\infty e^{-\lambda t} q_K(t, x, y) dt \quad (x, y \in \mathbb{R}^2);$$

to clarify, $q_K(t, x, y) = p(t, x - y) - r_K(t, x, y)$ with

$$r_K(t, x, y) := \mathbb{E}_x[p(t - \sigma_K, X(\sigma_K) - y) : \sigma_K < t]$$

as in [20] 2.5. We have that $g_K^\lambda f = R_Y(-\lambda)f$ m -a.e. on Y for $f \in L^2(Y)$. If $\lambda = 0$ the notation g_K is sometimes used.

As in [20] 3.4, the fundamental identities for logarithmic potentials read

$$k^\lambda = g_K^\lambda + h_K^\lambda k^\lambda - \langle \cdot, 1 \rangle W_K^\lambda, \quad (3.11)$$

$$k = g_K + h_K k - \langle \cdot, 1 \rangle W_K. \quad (3.12)$$

Lemma 3.1. *For any $\lambda, \mu \geq 0$ with $\lambda + \mu > 0$ we have that*

$$g_K^\lambda p_K^\mu = -(\mu - \lambda)^{-1} \{ p_K^\mu - p_K^\lambda \}.$$

Proof. The result follows as in the proof of [18] Proposition 4.13. \square

Let $T_r := \inf\{t > 0 : X_t \notin B(0, r)\}$ stand for the first exit time of $B(0, r)$. The notation $B(0, r)$ signifies the open ball with centre at the origin and radius $r > 0$.

Lemma 3.2. *For each $r > 0$,*

$$\mathbb{E}_0 [e^{-T_r}] = 1/I_0(r)$$

where I_0 stands for the modified Bessel function of order zero. Moreover,

$$\mathbb{E}_0 [e^{-T_r}] \sim (2\pi r)^{1/2} e^{-r}$$

as $r \rightarrow \infty$.

Proof. A direct computation leads to the identity. The asymptotic behaviour follows as in [25] 3.6. \square

Proposition 3.1. *We have that*

$$\langle W_K, p_K^1 \rangle = 1.$$

Proof. In a similar way to the proof of [20] Proposition 3.4.4, we write

$$\langle k(\cdot, y) - k(0, y) 1, p_K^1 \rangle = g_K p_K^1(y) + \langle h_K[k(\cdot, y) - k(0, y) 1], p_K^1 \rangle - \langle W_K, p_K^1 \rangle \quad (y \in \mathbb{R}^2) \quad (3.13)$$

The function

$$k(x, y) - k(0, y) = -\frac{1}{2\pi} \log \frac{|x - y|}{|y|} \quad (x \in \mathbb{R}^2 \setminus \{y\})$$

converges to zero uniformly on compacts as $y \rightarrow \infty$. From Lemma 3.1,

$$g_K p_K^1(y) = 1 - p_K^1(y).$$

This latter converges to 1 in the limit $y \rightarrow \infty$ by Lemma 3.2. Therefore, the expression on the right-hand side of (3.13) converges to $1 - \langle W_K, p_K^1 \rangle$.

We now show that the left-hand side vanishes in the limit. Let $\varepsilon > 0$ and choose $0 < \delta < 1$ with the property that

$$|\log(1 + \tau)| < 2\pi \varepsilon \text{ for any } \tau \in \mathbb{R} \text{ with } |\tau| < \delta.$$

For any $x, y \in \mathbb{R}^2$ with $|x| < \delta |y|$ we have

$$|k(x, y) - k(0, y)| < \varepsilon. \quad (3.14)$$

The left-hand side in (3.13) may be written,

$$\langle k(\cdot, y) - k(0, y) 1, p_K^1 \rangle = \int_{\mathbb{R}^2} \{k(x, y) - k(0, y)\} p_K^1(x) m(dx).$$

Decompose \mathbb{R}^2 into the disjoint union $\mathbb{R}^2 = F_y \dot{\cup} G_y$ with

$$F_y := \{x \in \mathbb{R}^2 : |x| < \delta |y|\} \text{ and } G_y := \{x \in \mathbb{R}^2 : |x| \geq \delta |y|\}.$$

The integral over F_y is bounded by $\varepsilon \text{Cap}_1(K)$ in modulus by (3.14). Re-write the integral over G_y as

$$-k(0, y) \int_{G_y} p_K^1(x) m(dx) + \int_{A_y} k(x, y) p_K^1(x) m(dx) + \int_{B_y} k(x, y) p_K^1(x) m(dx)$$

where

$$A_y := \{x \in \mathbb{R}^2 : |x| \geq \delta |y|, |x - y| \leq 1\} \text{ and } B_y := \{x \in \mathbb{R}^2 : |x| \geq \delta |y|, |x - y| > 1\}.$$

The first two integrals vanish in the limit by Lemma 3.2; for the last, use in addition the estimate

$$|\log |x - y|| \leq |x| + |y| \text{ for } |x - y| > 1.$$

□

Lemma 3.3. *Let $s > 1$. Then*

- (i) $W_K^\lambda \rightarrow W_K$ in \mathcal{H}_{-s} as $\lambda \downarrow 0$;
- (ii) $R_Y(-\lambda) \rightarrow R_Y(0)$ strongly in $B(\mathcal{H}_s, \mathcal{H}_{-s})$ as $\lambda \downarrow 0$.

Proof. From (3.11), (3.12) and Lemma 3.1 we have

$$\begin{aligned} k^\lambda p_K^1 &= \frac{1}{1-\lambda} \{p_K^\lambda - p_K^1\} + h_K^\lambda k^\lambda p_K^1 - C_1(K) W_K^\lambda, \\ k p_K^1 &= 1 - p_K^1 + h_K k p_K^1 - C_1(K) W_K. \end{aligned}$$

Consequently,

$$[k^\lambda - k] p_K^1 = \frac{1}{1-\lambda} \{p_K^\lambda - 1\} + \frac{\lambda}{1-\lambda} - \frac{\lambda}{1-\lambda} p_K^1 + h_K^\lambda [k^\lambda - k] p_K^1 + (h_K^\lambda - h_K) k p_K^1 - C_1(K) \{W_K^\lambda - W_K\}.$$

(i) follows with the help of Corollary 3.1 and the dominated convergence theorem.

From (3.11) resp. (3.12) it can be seen that g_K^λ resp. g_K map \mathcal{H}_s boundedly into \mathcal{H}_{-s} for any $s > 1$. Further,

$$g_K^\lambda - g_K = k^\lambda - k + h_K^\lambda (k^\lambda - k) + (h_K^\lambda - h_K) k - \langle \cdot, 1 \rangle \{W_K^\lambda - W_K\}.$$

The claim in (ii) now follows from Corollary 3.1, Theorem 3.1 and (i) above. We use the relation (3.6). □

Proposition 3.2. *It holds that*

$$\lim_{\lambda \downarrow 0} g^\lambda(u) \{1 - \langle W_K^\lambda, p_K^1 \rangle\} = R(K).$$

Proof. Applying (3.11) to the equilibrium measure μ_K we obtain the identity

$$k^\lambda \mu_K = h_K^\lambda k^\lambda \mu_K - W_K^\lambda$$

with the help of [20] Theorem 4.4.3 as μ_K is a probability measure with support in K^r . We derive

$$\langle W_K^\lambda, p_K^1 \rangle = \langle h_K^\lambda k^\lambda \mu_K, p_K^1 \rangle - \langle k^\lambda \mu_K, p_K^1 \rangle.$$

In virtue of (3.9) we have

$$\langle W_K, p_K^1 \rangle = R(K) \langle 1, p_K^1 \rangle - \langle k \mu_K, p_K^1 \rangle.$$

Using Proposition 3.1 we proceed,

$$\begin{aligned} 1 - \langle W_K^\lambda, p_K^1 \rangle &= R(K) \langle 1, p_K^1 \rangle - \langle h_K^\lambda [k^\lambda - k] \mu_K, p_K^1 \rangle - \langle h_K^\lambda k \mu_K, p_K^1 \rangle + \langle [k^\lambda - k] \mu_K, p_K^1 \rangle \\ &= \frac{R(K)}{g^\lambda(u)} \langle W_K^\lambda, p_K^1 \rangle - \langle h_K^\lambda [k^\lambda - k] \mu_K, p_K^1 \rangle + \langle [k^\lambda - k] \mu_K, p_K^1 \rangle. \end{aligned}$$

The result now follows from Lemma 3.3 (i), (3.6), (3.2), Theorem 3.1 (iii) and duality. □

4 Construction of inverse operators

Let X be a complex Banach space with dual space X' and duality pairing $\langle \cdot, \cdot \rangle$. Let $B(X)$ stand for the collection of bounded linear operators on X . The notation A^\times stands for the adjoint operator of $A \in B(X)$.

Lemma 4.1. *Assume that $A \in B(X)$ is bijective with inverse B . Let $y \in X$, $f \in X'$ and $\sigma \in \mathbb{C}$. Define A_σ to be the rank-one perturbation of A given by*

$$A_\sigma := A + \sigma \langle \cdot, f \rangle y.$$

Suppose that

$$\alpha := 1 + \sigma \langle By, f \rangle \neq 0. \quad (4.1)$$

Then A_σ is bijective and has inverse given by

$$B_\sigma = B - \alpha^{-1} \sigma \langle \cdot, B^\times f \rangle By.$$

Proof. Verify by direct computation that $A_\sigma B_\sigma = B_\sigma A_\sigma = I$. \square

Lemma 4.2. *Let $\delta > 0$. Suppose that $(A_\lambda)_{\lambda \in (0, \delta)}$ is a family of operators in $B(X)$. Assume that*

- (i) $A_\lambda \rightarrow A$ strongly as $\lambda \downarrow 0$ for some $A \in B(X)$;
- (ii) for each $\lambda \in (0, \delta)$, A_λ is bijective with inverse B_λ ;
- (iii) $B_\lambda \rightarrow B$ strongly as $\lambda \downarrow 0$ for some $B \in B(X)$.

Then A is bijective and has inverse B .

Proof. Given $x \in X$ write, for example,

$$B A x - x = (B - B_\lambda) A x + B_\lambda (A - A_\lambda) x.$$

By the uniform boundedness principle there exists a finite constant c such that $\|B_\lambda\| \leq c < \infty$ for all $\lambda \in (0, \delta)$. Take limits on the right-hand side using (i) and (iii) to see that $B A x - x = 0$. \square

Lemma 4.3. *For any $\lambda > 0$,*

- (i) $\left[I - R_Y^{(-1)}((1 - \lambda)^{-1})V \right] 1 = g^\lambda(u)^{-1} W_K^\lambda + \lambda p_K^\lambda$;
- (ii) $\left[I - V R_Y^{(-1)}((1 - \lambda)^{-1}) \right] p_K^1 = -V \left[g^\lambda(u)^{-1} W_K^\lambda + \lambda p_K^\lambda \right]$.

Proof. Item (ii) follows from (i) via the identity $p_K^1 = -V1$. This last follows from Lemma 3.1. Again by this lemma,

$$\begin{aligned} \left[I - R_Y^{(-1)}((1 - \lambda)^{-1})V \right] 1 &= 1 + R_Y^{(-1)}((1 - \lambda)^{-1})p_K^1 \\ &= 1 - (1 - \lambda) \{ I + (1 - \lambda) R_Y(-\lambda) \} p_K^1 \\ &= 1 - (1 - \lambda) p_K^1 - (1 - \lambda)^2 g_K^\lambda p_K^1 \\ &= 1 - (1 - \lambda) p_K^1 + (1 - \lambda) \{ p_K^1 - p_K^\lambda \} \\ &= g^\lambda(u)^{-1} W_K^\lambda + \lambda p_K^\lambda. \end{aligned}$$

\square

Define

$$\begin{aligned} A &:= I - [I + k] V, \\ A_\lambda &:= I + R^{(-1)}((1 - \lambda)^{-1})V - g^\lambda(u) \langle \cdot, p_K^1 \rangle 1 \quad (\lambda > 0). \end{aligned} \quad (4.2)$$

Proposition 4.1. *Let $s > 1$. Then*

- (i) $A \in B(\mathcal{H}_{-s})$;
- (ii) $A_\lambda \in B(\mathcal{H}_{-s})$ for any $\lambda > 0$;
- (iii) $A_\lambda \rightarrow A$ strongly in $B(\mathcal{H}_{-s})$ as $\lambda \downarrow 0$;
- (iv) $AW_K = R(K)1$.
- (v) $A^\times VW_K = -R(K)p_K^1$.

Proof. Statement (i) flows from [18] Theorem 2.1 and Lemma 3.1. From Theorem 2.1 and [18] Lemma 3.1, $R(-\lambda) \in B(\mathcal{H}_s, \mathcal{H}_{-s})$; (ii) now follows. For (iii), we may write

$$A_\lambda = I - (1 - \lambda) [I + (1 - \lambda)k^\lambda] V + \lambda(\lambda - 2)g^\lambda(u)\langle \cdot, p_K^1 \rangle 1$$

with the help of (3.5). Thus,

$$A_\lambda - A = \lambda V + [k - k^\lambda] V + \lambda(2 - \lambda) \{k^\lambda V - g^\lambda(u)\langle \cdot, p_K^1 \rangle 1\}.$$

The strong convergence follows from Corollary 3.1.

As for the identity (iv), from Lemma 3.1 and the first resolvent identity, we derive

$$\begin{aligned} Vp_K^\lambda &= (1 - \lambda)^{-1} \{p_K^\lambda - p_K^1\} - R(-1)p_K^\lambda, \\ R(-\lambda)Vp_K^\lambda &= (1 - \lambda)^{-1} \{R(-1)p_K^\lambda - R(-\lambda)p_K^1\}; \end{aligned}$$

the second flowing from the first. With their help, a computation leads to the identity

$$A_\lambda W_K^\lambda = g^\lambda(u) \{1 - \langle W_K^\lambda, p_K^1 \rangle\} 1 - \lambda g^\lambda(u) p_K^1 + \lambda(\lambda - 1)g^\lambda(u)R(-\lambda)p_K^1.$$

By (3.5) and (3.10),

$$\lambda(\lambda - 1)g^\lambda(u)R(-\lambda)p_K^1 = \lambda(\lambda - 1)g^\lambda(u) \{k^\lambda p_K^1 + g^\lambda(u)C_1(K)1\} \rightarrow 0 \text{ in } \mathcal{H}_{-s} \text{ as } \lambda \downarrow 0$$

by Corollary 3.1. This shows that

$$A_\lambda W_K^\lambda \rightarrow R(K)1 \text{ in } \mathcal{H}_{-s} \text{ as } \lambda \downarrow 0 \quad (4.3)$$

by Proposition 3.2. Finally, write

$$AW_K - R(K)1 = (A - A_\lambda)W_K + A_\lambda(W_K - W_K^\lambda) + A_\lambda W_K^\lambda - R(K)1$$

and use (4.3), Lemma 3.3 (i), and (iii). Of course, the family $(A_\lambda)_{\lambda \in (0,1)}$ is bounded by the uniform boundedness principle.

Lastly, $A^\times VW_K = VAW_K = -R(K)p_K^1$ by (iv). \square

Lemma 4.4. *Let $s > 1$. Assume that $R(K) \neq 0$. Then there exists $\delta > 0$ such that for each $\lambda \in (0, \delta)$ the operator A_λ is bijective with inverse B_λ given by*

$$B_\lambda = I - R_Y^{(-1)}((1 - \lambda)^{-1}V - \frac{1}{g^\lambda(u)\alpha_\lambda} \langle \cdot, V[W_K^\lambda + \lambda g^\lambda(u)p_K^\lambda] \rangle (W_K^\lambda + \lambda g^\lambda(u)p_K^\lambda)) \quad (4.4)$$

where

$$\alpha_\lambda = 1 - \langle W_K^\lambda, p_K^1 \rangle + \lambda g^\lambda(u) \langle p_K^\lambda, p_K^1 \rangle.$$

Proof. The counterpart α_λ of (4.1) reads

$$\alpha_\lambda := 1 - g^\lambda(u) \langle [I - R_Y^{(-1)}((1-\lambda)^{-1})V] 1, p_K^1 \rangle = 1 - \langle W_K^\lambda, p_K^1 \rangle + \lambda g^\lambda(u) \langle p_K^\lambda, p_K^1 \rangle$$

after simplification using Lemma 4.3. By Proposition 3.2, $g^\lambda(u)\alpha_\lambda \rightarrow R(K)$ as $\lambda \downarrow 0$. Consequently, there exists $\delta > 0$ such that $\alpha_\lambda \neq 0$ for $\lambda \in (0, \delta)$. By Lemma 4.1, A_λ is bijective with inverse as in (4.4); again, after making use of Lemma 4.3. \square

Define

$$B_0 := I + [I + R_Y(0)]V.$$

Lemma 4.5. *The following identities hold:*

- (i) $B_0 1 = 0$;
- (ii) $B_0^\times p_K^1 = 0$.

Proof. By Lemma 3.1,

$$B_0 1 = 1 + [I + R_Y(0)]V 1 = 1 - [I + R_Y(0)]p_K^1 = 1 - p_K^1 - \{1 - p_K^1\} = 0$$

giving (i). For (ii), $B_0^\times p_K^1 = -B_0^\times V 1 = -V B_0 1 = 0$. \square

In case $R(K) \neq 0$, define

$$B := B_0 - R(K)^{-1} \langle \cdot, V W_K \rangle W_K. \quad (4.5)$$

Proposition 4.2. *Let $s > 1$. Assume that $R(K) \neq 0$. Then*

- (i) $B_\lambda \rightarrow B$ strongly in $B(\mathcal{H}_{-s})$ as $\lambda \downarrow 0$;
- (ii) A is bijective in $B(\mathcal{H}_{-s})$ with inverse B as in (4.5);
- (iii) $B 1 = R(K)^{-1} W_K$;
- (iv) $B^\times = I + V [I + R_Y(0)] - R(K)^{-1} \langle \cdot, W_K \rangle V W_K$;
- (v) $B^\times p_K^1 = -R(K)^{-1} V W_K$.

Proof. (i) follows from Proposition 3.2 and Lemma 3.3. This together with Lemmas 4.2 and 4.4, and Proposition 4.1 (iii) yield (ii). To see (iii) use Proposition 4.1 (iv). For (v) use the identity $B^\times V = V B$ and (iii). \square

Lemma 4.6. *Let $s > 1$. Set*

$$\begin{aligned} \mathcal{M} &:= \{u \in \mathcal{H}_{-s} : \langle u, p_K^1 \rangle = 0\}, \\ \mathcal{W} &:= \{u \in \mathcal{H}_{-s} : \langle u, V W_K \rangle = 0\}. \end{aligned}$$

Then

- (i) $B_0 A = I$ on \mathcal{M} ;
- (ii) $A B_0 = I$ on \mathcal{W} .

Proof. Note that $A_\lambda = I + R^{(-1)}((1-\lambda)^{-1})V$ on \mathcal{M} for each $\lambda > 0$. Set

$$B_{0,\lambda} := I - R_Y^{(-1)}((1-\lambda)^{-1})V$$

on \mathcal{H}_{-s} . Then $B_{0,\lambda} \rightarrow B_0$ strongly in $B(\mathcal{H}_{-s})$ as $\lambda \downarrow 0$. By the second resolvent identity and density of $\mathcal{H} \cap \mathcal{M}$ in \mathcal{M} , $B_{0,\lambda} A_\lambda = I$ on \mathcal{M} . By Proposition 4.1 (iii) and as in Lemma 4.2 we obtain $B_0 A = I$ on \mathcal{M} . This establishes (i).

From Lemma 4.3 (ii), $B_{0,\lambda}^\times p_K^1 = -g^\lambda(u)^{-1} V W_K^\lambda - \lambda V p_K^\lambda$. Hence,

$$g^\lambda(u) \langle B_{0,\lambda} u, p_K^1 \rangle = \langle u, -V W_K^\lambda - \lambda g^\lambda(u) V p_K^\lambda \rangle \rightarrow -\langle u, V W_K \rangle = 0 \text{ as } \lambda \downarrow 0$$

for $u \in \mathcal{W}$. Therefore, $A_\lambda B_{0,\lambda} u = u - g^\lambda(u) \langle B_{0,\lambda} u, p_K^1 \rangle \rightarrow u$ as $\lambda \downarrow 0$. Now use strong convergence to obtain (ii). \square

5 A lattice-point counting lemma

We require a simple lattice-point counting lemma. Let us make the following definitions. For $n \in \mathbb{N}$ and $\mathbb{Z} \ni k < 0$ set

$$\begin{aligned} A(n, k) &:= \left\{ x \in \mathbb{Z}^n : x_j \leq 1 \text{ for } j = 1, 2, \dots, n \text{ and } \sum_{j=1}^n x_j = k \right\}; \\ a(n, k) &:= \text{Card}(A(n, k)). \end{aligned}$$

Lemma 5.1. *For $n \in \mathbb{N}$ and $\mathbb{Z} \ni k < 0$, it holds that*

$$a(n, k) \leq a(n) \{ |k| + (3/2)n \}^{n-1}.$$

The constant $a(n)$ is given by

$$a(n) = \frac{\sqrt{n}}{\alpha(n-1)} \frac{2^{n-1}}{(n-1)!}.$$

Here, $\alpha(n)$ stands for the volume of the unit ball $B(0, 1)$ in \mathbb{R}^n ; it is understood that $\alpha(0) = 1$.

Proof. First, notice that $a(1, k) = 1$. For $n = 2, 3, \dots$ and $r \leq (3/2)n$ set

$$\begin{aligned} H(n, r) &:= \left\{ x \in \mathbb{R}^n : x_j \leq 3/2 \text{ for } j = 1, 2, \dots, n \text{ and } \sum_{j=1}^n x_j = r \right\}, \\ h(n, r) &:= \sigma(H(n, r)), \end{aligned}$$

where σ stands for surface area measure. We claim that

$$h(n, r) = \frac{\sqrt{n}}{(n-1)!} \{ |r| + (3/2)n \}^{n-1} \quad (5.1)$$

for $r < 0$. To see this, introduce the set

$$S(n, r) := \left\{ x \in \mathbb{R}^n : x_j \geq 0 \text{ for } j = 1, 2, \dots, n \text{ and } \sum_{j=1}^n x_j = r \right\}$$

for $r \geq 0$ and $n \geq 2$. As in [27] (for example) its surface area is given by

$$s(n, r) := \sigma(S(n, r)) = \sqrt{n} r^{n-1} / (n-1)!.$$

Since

$$H(n, r) = (3/2)(1, \dots, 1) - S(n, -r + (3/2)n),$$

the formula (5.1) follows.

To prove the lemma, note the inclusion

$$\dot{\bigcup}_{x \in A(n, k)} B(x, 1/2) \cap H(n, k) \subseteq H(n, k)$$

where the left-hand side is a disjoint union. Computing surface area using (5.1) yields the claim. \square

6 Asymptotics of the spectral shift function

The result below follows from Theorem 2.1; the method of proof is similar to that used in the proof of [18] Lemma 4.7.

Lemma 6.1. *Let $l \in \mathbb{N}_0$ and $s > 2l + 1$. Then for $\zeta \in \mathbb{C} \setminus [0, \infty)$,*

$$\left\| R^{(-1)}((1 + \zeta)^{-1}) - \sum_{j=0}^l \sum_{k=0}^1 \zeta^j \eta^k A_j^k \right\|_{B(\mathcal{H}_s, \mathcal{H}_{-s})} = o(|\zeta|^l)$$

as $\zeta \rightarrow 0$. The coefficients are given by

$$\begin{aligned} A_0^1 &= -K_0^1, \\ A_0^0 &= -I - K_0^0, \\ A_1^1 &= -2K_0^1 - K_1^1, \\ A_1^0 &= -I - 2K_0^0 - K_1^0, \\ A_j^k &= -K_j^k - 2K_{j-1}^k - K_{j-2}^k \text{ for } j \geq 2 \text{ and } k \in \{0, 1\}. \end{aligned}$$

In the context of the last Lemma, we may write

$$I + R^{(-1)}((1 + \zeta)^{-1})V = I - [I + k] V + \sigma \langle \cdot, p_K^1 \rangle 1 + \sum_{j=1}^l \sum_{k=0}^1 \zeta^j \eta^k A_j^k V + o(|\zeta|^l) \quad (6.1)$$

in $B(\mathcal{H}_{-s})$ as $\mathbb{C} \setminus [0, \infty) \ni \zeta \rightarrow 0$. We use the shorthand

$$\sigma = a + b\eta$$

where $a = a_0$ and $b = c_0$. Define

$$A_\sigma := A + \sigma \langle \cdot, p_K^1 \rangle 1$$

with A as in (4.2).

Proposition 6.1. *Let $s > 1$. Then for small $\zeta \in \mathbb{C} \setminus [0, \infty)$, $A_\sigma \in B(\mathcal{H}_{-s})$ is bijective with inverse given by*

$$B_\sigma := B_0 + \sum_{k=-\infty}^{-1} \theta_k \eta^k \langle \cdot, VW_K \rangle W_K \quad (6.2)$$

where

$$\theta_k := (-1)^{-k} (1/b) \left(\frac{R(K) + a}{b} \right)^{-(k+1)} \text{ for } k = -1, -2, \dots \quad (6.3)$$

Moreover, there exists a finite constant c such that

$$\|B_\sigma\|_{B(\mathcal{H}_{-s})} \leq c < \infty$$

for small $\zeta \in \mathbb{C} \setminus [0, \infty)$.

For later use, we introduce the quantity

$$\theta := \left| \frac{R(K) + a}{b} \right|.$$

Note that θ is invertible; in fact, $\theta \geq \pi$ for all values of $R(K) \in \mathbb{R}$.

Proof. We first treat the case $R := R(K) \neq 0$. By Lemma 4.2 (iii) and Proposition 3.1,

$$\alpha_\sigma := 1 + \sigma \langle B1, p_K^1 \rangle = 1 + \sigma/R.$$

The above quantity is non-zero for small $\zeta \in \mathbb{C} \setminus [0, \infty)$. By Lemma 4.1 and Lemma 4.2 (ii), A_σ is bijective with inverse

$$B_\sigma = B - \frac{\sigma}{\alpha_\sigma} \langle \cdot, B^\times p_K^1 \rangle B1 = B + \frac{\sigma}{R(R + \sigma)} \langle \cdot, VW_K \rangle W_K$$

after simplifying using Lemma 4.2 (iii) and (v). Now use

$$\frac{\sigma}{R(R + \sigma)} = \frac{1}{R} - \frac{1}{R + \sigma} = \frac{1}{R} + \sum_{k=-\infty}^{-1} \theta_k \eta^k$$

with θ_k as in (6.3). The expression (4.5) for B leads to the result.

Now assume that $R = 0$. In this case, B_σ in (6.3) becomes

$$B_\sigma = B_0 - \frac{1}{\sigma} \langle \cdot, VW_K \rangle W_K.$$

For $u \in \mathcal{H}_{-s}$,

$$\begin{aligned} B_\sigma A_\sigma u &= B_0 A u + \sigma \langle u, p_K^1 \rangle B_0 1 - \frac{1}{\sigma} \langle A u, VW_K \rangle W_K - \langle u, p_K^1 \rangle \langle 1, VW_K \rangle W_K \\ &= B_0 A u + \langle u, p_K^1 \rangle W_K \end{aligned}$$

by Lemma 4.5 (i) and Proposition 4.1 (v). Each $u \in \mathcal{H}_{-s}$ may be written uniquely in the form $u = v + \alpha W_K$ for some $v \in \mathcal{M}$ and $\alpha \in \mathbb{C}$. As $AW_K = 0$ by Proposition 4.1 (iv), we obtain

$$B_\sigma A_\sigma u = B_0 A v + \alpha W_K = v + \alpha W_K = u$$

by Lemma 4.6 (i). On the other hand,

$$\begin{aligned} A_\sigma B_\sigma u &= AB_0 u + \sigma \langle B_0 u, p_K^1 \rangle 1 - \frac{1}{\sigma} \langle u, VW_K \rangle AW_K - \langle u, VW_K \rangle \langle W_K, p_K^1 \rangle 1 \\ &= AB_0 u - \langle u, VW_K \rangle 1 \end{aligned}$$

by Lemma 4.5 (ii) and Proposition 4.1 (iv). Each $u \in \mathcal{H}_{-s}$ may be written uniquely in the form $u = w + \beta 1$ for some $w \in \mathcal{W}$ and $\beta \in \mathbb{C}$. So

$$A_\sigma B_\sigma u = AB_0 w - \beta \langle 1, VW_K \rangle 1 = w + \beta 1 = u$$

by Lemma 4.5 (i) and Lemma 4.6 (ii). This shows that B_σ is the inverse of A_σ in the case $R = 0$.

The final claim follows from the fact that $\sum_{k=-\infty}^{-1} \theta_k \eta^k = -\frac{1}{R+\sigma}$ is bounded for small ζ . \square

Lemma 6.2. *Let $l \in \mathbb{N}_0$ and $s > 2l + 1$. Then for small $\lambda > 0$,*

$$I + \sum_{j=1}^l \sum_{k=0}^1 \lambda^j \eta^k B_\sigma A_j^k V = I - \sum_{j=1}^l \sum_{k=-\infty}^1 \lambda^j \eta^k E_j^k \quad (6.4)$$

in $B(\mathcal{H}_{-s})$. The coefficients E_j^k are given by

$$\begin{aligned} E_j^1 &= -B_0 A_j^1 V && \text{for } j = 1, 2, \dots, \\ E_j^0 &= -B_0 A_j^0 V - \theta_{-1} \langle \cdot, V A_j^1 VW_K \rangle W_K && \text{for } j = 1, 2, \dots, \\ E_j^k &= -\langle \cdot, V[\theta_k A_j^0 + \theta_{k+1} A_j^1] VW_K \rangle W_K && \text{for } j = 1, 2, \dots \text{ and } k = -1, -2, \dots \end{aligned}$$

The double-summation in (6.4) converges absolutely in norm.

Proof. Replace the expression for B_σ as in Proposition 6.1 to obtain

$$\begin{aligned}
I + \sum_{j=1}^l \sum_{k=0}^1 \lambda^j \eta^k B_\sigma A_j^k V &= I + \sum_{j=1}^l \sum_{k=0}^1 \lambda^j \eta^k \left\{ B_0 + \sum_{p=-\infty}^{-1} \theta_p \eta^p \langle \cdot, VW_K \rangle W_K \right\} A_j^k V \\
&= I + \sum_{j=1}^l \sum_{k=0}^1 \lambda^j \eta^k B_0 A_j^k V + \sum_{j=1}^l \sum_{k=0}^1 \sum_{p=-\infty}^{-1} \lambda^j \eta^{k+p} \theta_p \langle \cdot, VA_j^k VW_K \rangle W_K \\
&= I + \sum_{j=1}^l \sum_{k=0}^1 \lambda^j \eta^k B_0 A_j^k V + \sum_{j=1}^l \sum_{p=-\infty}^{-1} \lambda^j \eta^p \theta_p \langle \cdot, VA_j^0 VW_K \rangle W_K + \sum_{j=1}^l \sum_{p=-\infty}^{-1} \lambda^j \eta^{1+p} \theta_p \langle \cdot, VA_j^1 VW_K \rangle W_K \\
&= I + \sum_{j=1}^l \sum_{k=0}^1 \lambda^j \eta^k B_0 A_j^k V \\
&\quad + \sum_{j=1}^l \lambda^j \theta_{-1} \langle \cdot, VA_j^1 VW_K \rangle W_K \\
&\quad + \sum_{j=1}^l \sum_{k=-\infty}^{-1} \lambda^j \eta^k \{ \theta_k \langle \cdot, VA_j^0 VW_K \rangle W_K + \theta_{k-1} \langle \cdot, VA_j^1 VW_K \rangle W_K \} \\
&= I + \sum_{j=1}^l \lambda^j \eta B_0 A_j^1 V + \sum_{j=1}^l \lambda^j B_0 A_j^0 V \\
&\quad + \sum_{j=1}^l \lambda^j \theta_{-1} \langle \cdot, VA_j^1 VW_K \rangle W_K \\
&\quad + \sum_{j=1}^l \sum_{k=-\infty}^{-1} \lambda^j \eta^k \langle \cdot, V [\theta_k A_j^0 + \theta_{k-1} A_j^1] VW_K \rangle W_K.
\end{aligned}$$

From (6.3) it can be seen that there exist finite constants e_j ($j = 1, 2, \dots$) such that

$$\|E_j^k\|_{B(\mathcal{H}_{-s})} \leq e_j \theta^{-k} \quad (6.5)$$

for $k = \dots, -2, -1$. In fact, an estimate of the above form also extends to the case $k = 0$ and $k = 1$. This shows that the double-summation converges absolutely in norm. \square

Lemma 6.3. *Let $l \in \mathbb{N}_0$ and $s > 2l + 1$. Then*

$$\left\{ I - \sum_{j=1}^l \sum_{k=-\infty}^1 \lambda^j \eta^k E_j^k - o(\lambda^l) \right\}^{-1} = I + \sum_{j=1}^l \sum_{k=-\infty}^j \lambda^j \eta^k D_j^k + o(\lambda^l) \quad (6.6)$$

in $B(\mathcal{H}_{-s})$ as $\lambda \downarrow 0$. The coefficients D_j^k are given by

$$D_j^k = \sum_{|\alpha|=j, |\beta|=k} E_\alpha^\beta$$

where the multi-indices (α, β) belong to the set

$$(\alpha, \beta) \in \bigcup_{n=1}^{\infty} \mathbb{N}^n \times \Lambda^n$$

where $\Lambda := \{\dots, -2, -1, 0, 1\}$. The double-summation in (6.6) converges absolutely in norm.

Proof. The operator

$$T := \sum_{j=1}^l \sum_{k=-\infty}^1 \lambda^j \eta^k E_j^k + o(\lambda^l)$$

satisfies $\|T\|_{B(\mathcal{H}_{-s})} < 1$ for small $\lambda > 0$. The inverse of $I - T$ may expressed as a Neumann series with coefficients as stated.

Suppose that $j \in \mathbb{N}$ and $\mathbb{Z} \ni k < 0$. Using (6.5),

$$\|D_j^k\|_{B(\mathcal{H}_{-s})} \leq \sum_{n=1}^j \sum_{|\alpha|=j} \sum_{|\beta|=k} \|E_\alpha^\beta\|_{B(\mathcal{H}_{-s})} \leq \theta^{-k} \sum_{n=1}^j \sum_{|\alpha|=j} e_\alpha a(n, k).$$

Assume that $k \leq -(3/2)j$. By Lemma 5.1, the right-hand side may be estimated via

$$\left\{ \sum_{n=1}^j \sum_{|\alpha|=j} 2^{n-1} a(n) e_\alpha \right\} |k|^{j-1} \theta^{-k}.$$

The index n refers to the length of the multi-index α . An inequality of the above form can be extended to the case $k < 0$. In summary (for future use), for any $j \in \mathbb{N}$ and $k < 0$,

$$\|D_j^k\|_{B(\mathcal{H}_{-s})} \leq d_j |k|^{j-1} \theta^{-k} \quad (6.7)$$

for some finite constant d_j . This shows that the double-summation in (6.6) converges absolutely in norm. \square

Lemma 6.4. *Let $l \in \mathbb{N}_0$ and $s > 2l + 1$. Then*

$$\left(I + R^{(-1)}(\mu - \imath 0)V \right)^{-1} = \sum_{j=0}^l \sum_{k=-\infty}^j \lambda^j \eta^k B_j^k + o(\lambda^l) \quad (6.8)$$

in $B(\mathcal{H}_{-s})$ as $\lambda \downarrow 0$. The coefficients are given by

$$\begin{aligned} B_0^0 &= B_0, \\ B_0^k &= \theta_k \langle \cdot, VW_K \rangle W_K && \text{for } k = -1, -2, \dots, \\ B_j^j &= D_j^j B_0 && \text{for } j = 1, 2, \dots, \\ B_j^k &= D_j^k B_0 + \sum_{p+q=k} \theta_p \langle \cdot, VW_K \rangle D_j^q W_K && \text{for } j = 1, 2, \dots \text{ and } k < j. \end{aligned} \quad (6.9)$$

The double-summation in (6.8) converges absolutely in norm for small $\lambda > 0$.

Proof. We rewrite (6.1) using Lemma 6.2 as

$$I + R^{(-1)}(\mu - \imath 0)V = A_\sigma \left\{ I - \sum_{j=1}^l \sum_{k=-\infty}^1 \lambda^j \eta^k E_j^k - o(\lambda^l) \right\}.$$

Inverting using Lemmas 6.3 and 6.1 we obtain

$$\begin{aligned}
\left(I + R^{(-1)}(\mu - \imath 0)V \right)^{-1} &= B_\sigma + \sum_{j=1}^l \sum_{k=-\infty}^j \lambda^j \eta^k D_j^k B_\sigma + o(\lambda^l) \\
&= B_0 + \sum_{k=-\infty}^{-1} \theta_k \eta^k \langle \cdot, VW_K \rangle W_K \\
&\quad + \sum_{j=1}^l \sum_{k=-\infty}^j \lambda^j \eta^k D_j^k B_0 \\
&\quad + \sum_{j=1}^l \sum_{k=-\infty}^j \sum_{r=-\infty}^{-1} \lambda^j \eta^{k+r} \theta_r \langle \cdot, VW_K \rangle D_j^k W_K + o(\lambda^l) \\
&= B_0 + \sum_{k=-\infty}^{-1} \theta_k \eta^k \langle \cdot, VW_K \rangle W_K \\
&\quad + \sum_{j=1}^l \lambda^j \eta^j D_j^j B_0 \\
&\quad + \sum_{j=1}^l \sum_{k=-\infty}^{j-1} \lambda^j \eta^k \left\{ D_j^k B_0 + \sum_{p+q=k} \theta_p \langle \cdot, VW_K \rangle D_j^q W_K \right\} + o(\lambda^l).
\end{aligned}$$

For absolute convergence of the double-summation, let us first consider the term

$$\sum_{p+q=k, q < 0} \theta_p \langle \cdot, VW_K \rangle D_j^q W_K$$

for $j \in \mathbb{N}$ and $k \leq -2$. From (6.7) its norm may be estimated by

$$c \sum_{p+q=k, q < 0} |\theta_p| \|D_j^q\|_{B(\mathcal{H}_{-s})} \leq c' \left\{ \sum_{p+q=k, q < 0} |q|^{j-1} \right\} \theta^{-k} \leq c' |k|^j \theta^{-k}$$

Consequently, for any $j \in \mathbb{N}_0$ and $k < 0$,

$$\|B_j^k\|_{B(\mathcal{H}_{-s})} \leq b_j |k|^j \theta^{-k} \quad (6.10)$$

for some finite constant b_j . So the double-summation converges absolutely in norm. \square

Proposition 6.2. *Let $l \in \mathbb{N}_0$. There exist $T_j^k \in \mathfrak{S}_1(\mathfrak{h})$, $0 \leq j \leq 2l$, $-\infty < 2k \leq j$ such that*

$$T(\lambda) = \sum_{0 \leq j \leq 2l} \sum_{-\infty < 2k \leq j} (\imath \lambda^{1/2})^j \eta^k T_j^k + o(\lambda^l) \quad (6.11)$$

in $\mathfrak{S}_1(\mathfrak{h})$ as $\lambda \downarrow 0$. The coefficients are given by

$$T_j^k = 2\pi \imath \sum_{p+2q+r=j} (-1)^{q+r} U_p^{(-1)} V B_q^k U_r^{(-1)*} \quad (6.12)$$

for $0 \leq j \leq 2l$ and $-\infty < 2k \leq j$. The double-summation in (6.11) converges absolutely in norm. Also,

$$T_0^0 = 0.$$

Proof. The argument proceeds as in [18] Proposition 4.4. Choose $s > 2l + 1$. As in (2.9),

$$U^{(-1)}(\mu) = \sum_{p=0}^{2l} (\imath \lambda^{1/2})^p U_p^{(-1)} + o(\lambda^l)$$

in $\mathfrak{S}_2(\mathcal{H}_s, \mathfrak{h})$ as $\lambda \downarrow 0$. By Lemma 6.4,

$$\left(I + R^{(-1)}(\mu - \imath 0)V \right)^{-1} = \sum_{q=0}^l \sum_{k=-\infty}^q (-1)^q (\imath \lambda^{1/2})^{2q} \eta^k B_q^k + o(\lambda^l)$$

in $B(\mathcal{H}_{-s})$ as $\lambda \downarrow 0$. The expansion follows straightforwardly.

Fix $j \in \mathbb{N}_0$ and $k < 0$. By (6.10),

$$\|T_j^k\|_{\mathfrak{S}_1(\mathfrak{h})} \leq c \sum_{p+2q+r=j} \|B_q^k\|_{B(\mathcal{H}_{-s})} \leq c \sum_{p+2q+r=j} b_q |k|^q \theta^{-k} \leq c' \left\{ \sum_{p+2q+r=j} 1 \right\} |k|^{[j/2]} \theta^{-k}$$

Thus for each $j \in \mathbb{N}_0$, there exists a finite constant t_j such that

$$\|T_j^k\|_{\mathfrak{S}_1(\mathfrak{h})} \leq t_j \langle k \rangle^{[j/2]} \theta^{-k} \quad (6.13)$$

for $-\infty < 2k \leq j$. This establishes the summability claim.

From (6.12), (2.11) and (2.7),

$$T_0^0 = 2\pi \imath U_0 V B_0 U_0^* = (\imath/4\pi) \langle V B_0 1, 1 \rangle \langle \cdot, 1 \rangle.$$

By Lemma 4.5 (i), $B_0 1 = 0$; hence $T_0^0 = 0$. □

Theorem 6.1. *Let $l \in \mathbb{N}$. Then*

$$\xi(\lambda) = \sum_{k=-l}^{-1} \xi_0^k \eta^k + O(\eta^{-(l+1)})$$

as $\lambda \downarrow 0$ where the coefficients are given by

$$\xi_0^k = \frac{1}{2\pi \imath} \sum_{|\alpha|=0, |\beta|=k} \frac{(-1)^p}{p} \text{Tr} [T_\alpha^\beta]. \quad (6.14)$$

In the above, p signifies the length of the multi-index α (resp. β).

Proof. For small $\lambda > 0$,

$$\xi(\lambda) = \frac{-1}{2\pi \imath} \text{Tr} \log(I + T(\lambda))$$

in virtue of (2.17). From Proposition 6.2 we extract the expansion

$$T(\lambda) = \sum_{k=-l}^{-1} \eta^k T_0^k + O(\eta^{-(l+1)})$$

and insert into the formula

$$\log(I + T) = \sum_{p=1}^{\infty} \frac{(-1)^{p+1}}{p} T^p$$

valid for $T \in B(\mathfrak{h})$ with $\|T\| < 1$. □

7 First three coefficients in low-energy expansion of the scattering phase

Lemma 7.1. *The following identities hold:*

(i)

$$\xi_0^{-1} = -\frac{1}{2\pi\imath} \text{Tr}[T_0^{-1}],$$

(ii)

$$\xi_0^{-2} = \frac{1}{2\pi\imath} \{ -\text{Tr}[T_0^{-2}] + (1/2) \text{Tr}[T_0^{-1}T_0^{-1}] \},$$

(iii)

$$\xi_0^{-3} = \frac{1}{2\pi\imath} \{ -\text{Tr}[T_0^{-3}] + (1/2) (\text{Tr}[T_0^{-2}T_0^{-1}] + \text{Tr}[T_0^{-1}T_0^{-2}]) - (1/3) \text{Tr}[T_0^{-1}T_0^{-1}T_0^{-1}] \}.$$

Proof. These expressions follow directly from (6.14). \square

Theorem 7.1. *The following identities hold:*

(i) $\xi_0^{-1} = 1;$

(ii) $\xi_0^{-2} = C(K) - \log 4 + 2\gamma;$

(iii) $\xi_0^{-3} = (C(K) - \log 4 + 2\gamma)^2 - \frac{\pi^2}{3}.$

Proof. First note that from (6.9) the identity $\langle 1, VB_0^k 1 \rangle = \theta_k$ holds for any $\mathbb{Z} \ni k < 0$. Moreover, from (6.12),

$$T_0^k = 2\pi\imath U_0 V B_0^k U_0^*$$

for any $k < 0$. With the help of [19] Corollary 7.2 (i) (or straightforwardly from (2.7)),

$$\begin{aligned} \text{Tr}[T_0^k] &= 2\pi\imath \text{Tr}[U_0 V B_0^k U_0^*] \\ &= 2\pi\imath (1/4\pi) \langle V B_0^k 1, 1 \rangle \\ &= (\imath/2) \theta_k. \end{aligned} \tag{7.1}$$

From (6.3) we have

$$\begin{aligned} \theta_{-1} &= -\frac{1}{b}, \\ \theta_{-2} &= \frac{1}{b} \left(\frac{R+a}{b} \right), \\ \theta_{-3} &= -\frac{1}{b} \left(\frac{R+a}{b} \right)^2 \end{aligned}$$

with

$$a = (1/2\pi) (\log 2 - \gamma) + \imath/4 \text{ and } b = 1/4\pi.$$

(i) From (7.1),

$$\text{Tr}[T_0^{-1}] = (\imath/2) \theta_{-1} = -2\pi\imath.$$

This and Lemma 7.1 (i) gives the first item.

(ii) Using the above identity once more,

$$\text{Tr}[T_0^{-2}] = (\imath/2) \theta_{-2}.$$

With the help of [19] Corollary 7.2 (vii),

$$\begin{aligned}\mathrm{Tr}[T_0^{-1}T_0^{-1}] &= (2\pi\imath)^2 \mathrm{Tr}[U_0 V B_0^{-1} U_0^* U_0 V B_0^{-1} U_0^*] \\ &= -(2\pi)^2 (1/4) (2\pi)^{-4} (2\pi)^2 \langle V B_0^{-1} 1, 1 \rangle^2 \\ &= -(1/4) \theta_{-1}^2.\end{aligned}$$

By Lemma 7.1 (ii),

$$\xi_0^{-2} = \frac{1}{2\pi\imath} \{ -(1/2) \imath \theta_{-2} - (1/8) \theta_{-1}^2 \} = -4\pi [R + \Re a] = C(K) - \log 4 + 2\gamma.$$

(iii) From (7.1),

$$\mathrm{Tr}[T_0^{-3}] = (\imath/2) \theta_{-3}.$$

By [19] Corollary 7.2 (vii),

$$\begin{aligned}\mathrm{Tr}[T_0^{-2}T_0^{-1}] &= (2\pi\imath)^2 \mathrm{Tr}[U_0 V B_0^{-2} U_0^* U_0 V B_0^{-1} U_0^*] \\ &= -(2\pi)^2 (1/4) (2\pi)^{-4} (2\pi)^2 \langle V B_0^{-2} 1, 1 \rangle \langle V B_0^{-1} 1, 1 \rangle \\ &= -(1/4) \theta_{-2} \theta_{-1}.\end{aligned}$$

By [19] Corollary 7.2 (viii),

$$\begin{aligned}\mathrm{Tr}[T_0^{-1}T_0^{-1}T_0^{-1}] &= (2\pi\imath)^3 \mathrm{Tr}[U_0 V B_0^{-1} U_0^* U_0 V B_0^{-1} U_0^* U_0 V B_0^{-1} U_0^*] \\ &= -\imath (2\pi)^3 (1/8) (2\pi)^{-6} (2\pi)^3 \langle V B_0^{-1} 1, 1 \rangle^3 \\ &= -(\imath/8) \theta_{-1}^3.\end{aligned}$$

By Lemma 7.1 (iii) and some computation,

$$\begin{aligned}\xi_0^{-3} &= \frac{1}{2\pi\imath} \{ -(\imath/2) \theta_{-3} - (1/4) \theta_{-2} \theta_{-1} - (1/3) (-\imath/8) \theta_{-1}^3 \} \\ &= (4\pi)^2 \left\{ (R + a - \imath/4)^2 - \frac{1}{48} \right\} \\ &= (C(K) - \log 4 + 2\gamma)^2 - \frac{\pi^2}{3}.\end{aligned}$$

□

8 Asymptotics of the pinned Wiener sausage

We first remark that $\gamma(t)$ may be written purely analytically as

$$\gamma(t) = (4\pi t) \mathrm{Tr} [e^{-tH} - e^{-tH_Y}]. \quad (8.1)$$

Let $0 < \delta < 1$. For $k \in \mathbb{Z}$,

$$\int_0^\delta t e^{-t\lambda} (-\log \lambda)^k d\lambda \sim \sum_{r=0}^\infty (-1)^r \binom{k}{r} \Gamma^{(r)}(1) (\log t)^{k-r} \quad (8.2)$$

as $t \rightarrow \infty$ according to [9] Lemma 3. Recall that for $k < 0$, the binomial is specified by

$$\binom{k}{r} = (-1)^r \binom{-k+r-1}{r}.$$

Theorem 8.1. *Let $l \in \mathbb{N}$. Then*

$$\gamma(t) = \sum_{k=-l}^{-1} \gamma_0^k t (\log t)^k + o(t (\log t)^{-l})$$

as $t \rightarrow \infty$ where

$$\gamma_0^k = 4\pi \sum_{s-r=k} \xi_0^s (-1)^r \binom{s}{r} \Gamma^{(r)}(1). \quad (8.3)$$

The extra constraints $-\infty < s \leq -1$ and $r \geq 0$ apply in the summation.

Proof. We write

$$\gamma(t) = (4\pi t) \left\{ \int_0^\delta te^{-t\lambda} \xi(\lambda) d\lambda + \int_\delta^\infty te^{-t\lambda} \xi(\lambda) d\lambda \right\}.$$

In virtue of (2.13) the second term decays exponentially. Write

$$\xi(\lambda) = \sum_{k=-l}^{-1} \xi_0^k \eta^k + O(\eta^{-(l+1)})$$

according to Theorem 6.1. By (8.2) the term

$$(4\pi t) \int_0^\delta te^{-t\lambda} \eta^{-(l+1)} d\lambda = o(t(\log t)^{-l})$$

can be absorbed into the remainder. Again by (8.2), for $-l \leq k \leq -1$,

$$(4\pi t) \int_0^\delta te^{-t\lambda} \eta^k d\lambda = 4\pi \sum_{r=0}^{k+l} (-1)^r \binom{k}{r} \Gamma^{(r)}(1) t(\log t)^{k-r} + o(t(\log t)^{-l})$$

as $t \rightarrow \infty$. Therefore,

$$\begin{aligned} (4\pi t) \int_0^\delta te^{-t\lambda} \sum_{k=-l}^{-1} \xi_0^k \eta^k d\lambda &= \sum_{k=-l}^{-1} \xi_0^k (4\pi t) \int_0^\delta te^{-t\lambda} \eta^k d\lambda \\ &= \sum_{k=-l}^{-1} \xi_0^k 4\pi \sum_{r=0}^{k+l} (-1)^r \binom{k}{r} \Gamma^{(r)}(1) t(\log t)^{k-r} + o(t(\log t)^{-l}) \\ &= \sum_{k=-l}^{-1} \left\{ 4\pi \sum_{s-r=k} (-1)^r \xi_0^s \binom{s}{r} \Gamma^{(r)}(1) \right\} t(\log t)^k + o(t(\log t)^{-l}). \end{aligned}$$

It is understood that $-\infty < s \leq -1$ and $r \geq 0$ in the summation. \square

Corollary 8.1. *The following identities hold:*

- (i) $\gamma_0^{-1} = 4\pi$;
- (ii) $\gamma_0^{-2} = 4\pi \{ C(K) + \gamma - \log 4 \}$;
- (iii) $\gamma_0^{-3} = 4\pi \left\{ (C(K) + \gamma - \log 4)^2 - \frac{\pi^2}{6} \right\}$.

Proof. From (8.3) we derive

- (a) $\gamma_0^{-1} = 4\pi \xi_0^{-1}$;
- (b) $\gamma_0^{-2} = 4\pi \{ \xi_0^{-1} \Gamma^{(1)}(1) + \xi_0^{-2} \}$;
- (c) $\gamma_0^{-3} = 4\pi \{ \xi_0^{-1} \Gamma^{(2)}(1) + 2\xi_0^{-2} \Gamma^{(1)}(1) + \xi_0^{-3} \}$.

According to [1] 6.4.2,

$$\Gamma^{(1)}(1) = -\gamma, \quad \Gamma^{(2)}(1) = \gamma^2 + \frac{\pi^2}{6}.$$

The identities (i)-(iii) now follow with the help of Theorem 7.1. \square

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9 Appendix

In this Appendix we prove Theorem 2.1, Lemma 2.1 and Theorem 2.2.

Lemma 9.1. *The operator K with convolution kernel*

$$k(x) := (\log |x|) |x|^\alpha \quad (-2 < \alpha < \infty)$$

belongs to $\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})$ whenever $s > \alpha \vee 0 + 1$.

Proof. Consider the operator K with convolution kernel $k(x) := |x|^\alpha$. Suppose that $\alpha \geq 0$. For $s > \alpha + 1$,

$$\begin{aligned} \|K\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})}^2 &= \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle x \rangle^{-2s} |x - y|^{2\alpha} \langle y \rangle^{-2s} dy dx \\ &\leq 4^\alpha \int_{\mathbb{R}^2 \times \mathbb{R}^2} \langle x \rangle^{-2s+2\alpha} \langle y \rangle^{-2s+2\alpha} dy dx < \infty. \end{aligned}$$

In case the kernel k includes the logarithmic term, split the integral into a sum of integrals over the domains

$$A_1 := \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : 0 < |x - y| < 1\} \text{ and } A_2 := \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x - y| > 1\}.$$

On A_1 use Young's inequality ([8] 1.1.4) and on A_2 use the inequality

$$\log |x - y| \leq 2^\varepsilon \varepsilon^{-1} \langle x \rangle^\varepsilon \langle y \rangle^\varepsilon \quad (9.1)$$

valid for any $\varepsilon > 0$. The above decomposition can also be used to deal with the case $-2 < \alpha < 0$. \square

Lemma 9.2. *Let $\beta > 2$. For $y \in \mathbb{R}^2$ and $0 < r \leq 1/2$ define*

$$f(y, r) := \int_{|x-y| \geq r^{-1}} \langle x \rangle^{-\beta} dx.$$

Then there exists a finite constant c such that

$$f(y, r) \leq \begin{cases} cr^{\beta-2} & \text{for } |y| \leq \frac{1}{2r}, \\ c & \text{for } |y| > \frac{1}{2r}. \end{cases}$$

Proof. The result for $|y| > 1/2r$ is clear. Suppose that $|y| \leq 1/2r$. Then $1 - r|y| \geq 1/2 \geq r$. Thus, $B(0, r^{-1} - |y|) \subseteq B(y, r^{-1})$ and $B(0, 1/2) \subseteq B(0, 1 - r|y|)$. This means that

$$f(y, r) \leq \int_{|x| \geq r^{-1} - |y|} \langle x \rangle^{-\beta} dx \leq r^{\beta-2} \int_{|x| \geq 1/2} |x|^{-\beta} dx.$$

\square

Let φ_1 denote the indicator function of the interval $[0, 1]$ and $\varphi_2 := 1 - \varphi_1$.

Lemma 9.3. *Let $K(\zeta)$ be the operator with convolution kernel*

$$k(x; \zeta) := \varphi_2(|\zeta|^{1/2} |x|) (\log |x|) |x|^\alpha \quad (\zeta \in \mathbb{C} \setminus [0, \infty), \alpha \in \mathbb{R}).$$

Let $s > \alpha \vee 0 + 1$. Then $K(\zeta)$ belongs to $\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})$ and

$$\|K(\zeta)\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})} = O(|\zeta|^{(s-\alpha-1)/2})$$

as $\zeta \rightarrow 0$.

Proof. Consider the operator $K(\zeta)$ with convolution kernel $k(x; \zeta) := \varphi_2(|\zeta|^{1/2}|x|)|x|^\alpha$. Suppose that $\alpha \geq 0$. For $s > \alpha + 1$ we find

$$\|K(\zeta)\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})}^2 \leq 4^\alpha \int_{\mathbb{R}^2} \langle y \rangle^{-\beta} f(y, |\zeta|^{1/2}) dy$$

where f is defined as in Lemma 9.2 and $\beta := 2(s - \alpha)$ and $r := |\zeta|^{1/2}$. Using the estimate in Lemma 9.2 this may be bounded by

$$4^\alpha c \left\{ |\zeta|^{s-\alpha-1} \int_{|y| \leq 1/2|\zeta|^{1/2}} \langle y \rangle^{-\beta} dy + \int_{|y| > 1/2|\zeta|^{1/2}} \langle y \rangle^{-\beta} dy \right\}$$

for $0 < |\zeta| \leq 1/4$. The latter integral has order $O(|\zeta|^{s-\alpha-1})$ as $\zeta \rightarrow 0$. This gives the result for $\alpha \geq 0$. Now suppose that $\alpha < 0$. For $s > 1$,

$$\|K(\zeta)\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})}^2 \leq |\zeta|^{-\alpha} \int_{|x-y| \geq |\zeta|^{-1/2}} \langle x \rangle^{-2s} \langle y \rangle^{-2s} dy dx.$$

Combining this with the result for $\alpha = 0$ yields the result for this case. In case the kernel $k(\cdot; \zeta)$ includes the logarithmic term, make use of (9.1). \square

Lemma 9.4. *Let $K(\zeta)$ be the operator with convolution kernel*

$$k(x; \zeta) := \varphi_1(|\zeta|^{1/2}|x|)|x|^\alpha \quad (\zeta \in \mathbb{C} \setminus [0, \infty), \alpha > 0).$$

Let $\alpha < s \leq \alpha + 1$ with $s > 1$. Then $K(\zeta)$ belongs to $\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})$ and

$$\|K(\zeta)\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})} = \begin{cases} O(|\zeta|^{(s-\alpha-1)/2}) & \text{if } \alpha < s < \alpha + 1, \\ O((-\log |\zeta|)^{1/2}) & \text{if } s = \alpha + 1, \end{cases}$$

as $\zeta \rightarrow 0$.

Proof. Using the fact that

$$|x - y|^{2\alpha} \leq 2^{2\alpha} \langle y \rangle^{2\alpha} \text{ for } |y| \geq |x|$$

we have

$$\begin{aligned} \|K(\zeta)\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})}^2 &= \int_{|x-y| \leq |\zeta|^{-1/2}} \langle x \rangle^{-2s} |x - y|^{2\alpha} \langle y \rangle^{-2s} dy dx \\ &\leq 2^{2\alpha+1} \int_{\mathbb{R}^2} \langle x \rangle^{-2s} \int_{|x-y| \leq |\zeta|^{-1/2}} \langle y \rangle^{2(\alpha-s)} dy dx \\ &\leq 2^{2\alpha+1} \int_{\mathbb{R}^2} \langle x \rangle^{-2s} \int_{|y| \leq |\zeta|^{-1/2}} \langle y \rangle^{2(\alpha-s)} dy dx. \end{aligned}$$

For $\alpha < s < \alpha + 1$,

$$\int_{|y| \leq |\zeta|^{-1/2}} \langle y \rangle^{2(\alpha-s)} dy \leq \frac{\pi}{\alpha - s + 1} 4^{\alpha-s+1} |\zeta|^{s-\alpha-1}$$

for $0 < |\zeta| < 1$. On the other hand, for $s = \alpha + 1$,

$$\int_{|y| \leq |\zeta|^{-1/2}} \langle y \rangle^{2(\alpha-s)} dy \leq 2\pi \left\{ -\frac{1}{2} \log |\zeta| + \log 2 \right\}$$

again for $0 < |\zeta| < 1$. This leads to the result. \square

Proof of Theorem 2.1. First recall that by [1] 9.2.3,

$$\left| H_0^{(1)}(z) \right| \leq c |z|^{-1/2} \text{ for } |z| > 1 \text{ and } 0 < \text{Arg } z < \pi. \quad (9.2)$$

With φ_1, φ_2 as before set

$$k^{(j)}(x; \zeta) := \varphi_j(|\zeta|^{1/2}|x|)k(x; \zeta) \quad (j = 1, 2)$$

with $k(\cdot; \zeta)$ as in (2.6). Using Lemma 9.1 and the estimate (9.2) for $k^{(2)}(x; \zeta)$ it may be seen that $R(\zeta)$ belongs to $\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})$ for any $s > 1$.

Let $l \in \mathbb{N}_0$ and $s > 2l + 1$. By Lemma 9.1 each of the operators K_j^ε belongs to $\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})$ for $j = 0, \dots, l$ and $\varepsilon = 0, 1$. Define

$$k_l(x; \zeta) := \sum_{j=0}^l \sum_{\varepsilon=0}^1 \zeta^j \eta^\varepsilon k_j^\varepsilon(x) \quad (x \in \mathbb{R}^2 \setminus \{0\})$$

and the cut-off kernels $k_l^{(j)}(\cdot; \zeta)$ ($j = 1, 2$) as above. By (2.2) there exists a finite constant c such that

$$\left| k^{(1)}(x; \zeta) - k_l^{(1)}(x; \zeta) \right| \leq c \varphi_1(|\zeta|^{1/2}|x|) |\zeta|^{l+1} |\eta| \left(1 + |\log |x|| \right) |x|^{l+1}$$

for small ζ . Set $\alpha = l + 1$. For $l \geq 1$ we have that $s > \alpha + 1$. The remainder estimate follows from Lemma 9.1. Consider the case $l = 0$. If $s > 2$ use Lemma 9.1. If $1 < s \leq 2$ use Lemma 9.4. To deal with the logarithmic term consider the operators with kernels

$$\varphi_j(|x|) \left\{ k^{(1)}(x; \zeta) - k_l^{(1)}(x; \zeta) \right\} \quad (j = 1, 2).$$

The operator corresponding to $j = 1$ is bounded by Young's inequality. For the second use the fact that for any $\varepsilon > 0$ there exists a finite constant c such that

$$\varphi_2(|x|) \log |x| \leq c |x|^\varepsilon \text{ for } x \in \mathbb{R}^2.$$

In view of (9.2) we have that

$$|k^{(2)}(x; \zeta)| \leq c |\zeta|^{-1/4} \varphi_2(|\zeta|^{1/2}|x|) |x|^{-1/2}$$

and by Lemma 9.3 we obtain that

$$\|K^{(2)}(\zeta)\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathcal{H}_{-s})} = O(|\zeta|^{\frac{s-1}{2}}) = o(|\zeta|^l).$$

Similar considerations can be used to deal with the terms in $K_l^{(2)}(\zeta)$. □

Proof of Lemma 2.1. The kernel of $U(\lambda)$ is given by

$$u(\omega, x; \lambda) = \frac{1}{\sqrt{2}} (2\pi)^{-1} e^{-i\lambda^{1/2}\omega \cdot x}.$$

Therefore, $u(\omega, x; \lambda)$ has an absolutely convergent series expansion of the form

$$u(\omega, x; \lambda) = \sum_{j=0}^{\infty} (i\lambda^{1/2})^j u_j(\omega, x)$$

where

$$u_j(\omega, x) := \frac{1}{\sqrt{2}} (2\pi)^{-1} \frac{(-1)^j}{j!} (\omega \cdot x)^j.$$

It is clear that $u(\omega, x; \lambda)$ is uniformly bounded. The truncated kernel will be written $u_l(\omega, x; \lambda)$. Let φ_1, φ_2 be as previously. Define

$$u^{(j)}(\omega, x; \lambda) := \varphi_j(\lambda^{1/2}|x|)u(\omega, x; \lambda) \quad (j = 1, 2)$$

and $u_l^{(j)}(\omega, x; \lambda)$ similarly. Denote the corresponding operators by $U^{(1)}(\lambda)$, etc. Let U be the operator with kernel $u(\omega, x) := |x|^j$ ($j \in \mathbb{N}_0$). Note that $U \in \mathfrak{S}_2(\mathcal{H}_s, \mathfrak{h})$ if $s > j + 1$. Let $U^{(j)}(\lambda)$ be the operator with kernel

$$u^{(j)}(\omega, x; \lambda) := \varphi_j(\lambda^{1/2}|x|)|x|^{2j}.$$

Then $U^{(1)}(\lambda)$ has Hilbert-Schmidt norm $O(\lambda^{(s-j-1)/2})$ provided $s \leq j + 1$. On the other hand, the operator $U^{(2)}(\lambda)$ has norm $O(\lambda^{(s-j-1)/2})$ if $s > j + 1$.

We have the estimate

$$|u^{(1)}(\omega, x; \lambda) - u_l^{(1)}(\omega, x; \lambda)| \leq c(\lambda^{1/2}|x|)^{l+1}.$$

Thus $\|U^{(1)}(\lambda) - U_l^{(1)}(\lambda)\|_{\mathfrak{S}_2(\mathcal{H}_s, \mathfrak{h})} = o(\lambda^{l/2})$ provided $s > l + 1$. It is straightforward to see that the Hilbert-Schmidt norm of the difference $U^{(2)}(\lambda) - U_l^{(2)}(\lambda)$ admits an estimate of the same order in λ . \square

Proof of Theorem 2.2. The compactness statement is equivalent to the result $\langle \cdot \rangle^s V \langle \cdot \rangle^s \in \mathfrak{S}_\infty(\mathcal{H})$. Define $V(t) := J e^{-tH_Y} J^* - e^{-tH}$. Then

$$\langle \cdot \rangle^s V(2t) \langle \cdot \rangle^s = \langle \cdot \rangle^s V(t) e^{-tH} \langle \cdot \rangle^s + \langle \cdot \rangle^s J e^{-tH_Y} J^* V(t) \langle \cdot \rangle^s.$$

The kernel $k(t; x, y)$ of $\langle \cdot \rangle^{-s} e^{-tH} \langle \cdot \rangle^s$ is well-known to be

$$k(t; x, y) = (4\pi t)^{-1} \langle x \rangle^{-s} e^{-|x-y|^2/4t} \langle y \rangle^s.$$

Using the inequality

$$\langle y \rangle^s \leq 2^s (\langle x \rangle^s + \langle x - y \rangle^s)$$

we see that $k(t; x, y)$ is dominated by a square-integrable convolution kernel and hence by Young's inequality ([8] 1.1.4 for example) $\langle \cdot \rangle^{-s} e^{-tH} \langle \cdot \rangle^s \in B(\mathcal{H}, L^\infty(\mathbb{R}^2))$. Let $M = (\Omega, \mathcal{M}, X_t, \mathbb{P}_x)$ be Brownian motion on \mathbb{R}^2 . Put $\sigma_K := \inf\{t > 0 : X_t \in K\}$, the first hitting time of K . By the strong Markov property of Brownian motion,

$$\begin{aligned} |\langle \cdot \rangle^s V(t) \langle \cdot \rangle^s 1(x)| &= \langle x \rangle^s \mathbb{E}_x(\langle X_t \rangle^s : \sigma(K) < t) \\ &= \langle x \rangle^s \mathbb{E}_x(\mathbb{E}_{X_{\sigma(K)}} \langle X_{t-\sigma(K)} \rangle^s : \sigma(K) < t). \end{aligned} \quad (9.3)$$

Since

$$\sup_{y \in K} \sup_{0 \leq \tau \leq t} \mathbb{E}_y \langle X_\tau \rangle^s$$

is finite, (9.3) is square-integrable. Therefore, $\langle \cdot \rangle^s V(t) \langle \cdot \rangle^s \in B(L^\infty(\mathbb{R}^2), \mathcal{H})$. We conclude that $\langle \cdot \rangle^s V(t) e^{-tH} \langle \cdot \rangle^s \in \mathfrak{S}_2(\mathcal{H})$ (see [24] for example) and hence the same for $\langle \cdot \rangle^s V(2t) \langle \cdot \rangle^s$ by domination and duality.

Compactness of $\langle \cdot \rangle^s V \langle \cdot \rangle^s$ follows once we have shown that

$$\int_0^\infty e^{-t} \|\langle \cdot \rangle^s V \langle \cdot \rangle^s\|_{B(\mathcal{H})} dt < \infty$$

by [29] Theorem 1.3 and Remark 1.2 (b). Applying Hölder's inequality inside the functional integral we obtain for any $f \in \mathcal{H}$,

$$\|\langle \cdot \rangle^s V(t) \langle \cdot \rangle^s f\|_2 \leq \sup_{y \in K} \sup_{0 \leq \tau \leq t} (\mathbb{E}_y \langle X_\tau \rangle^{2s})^{1/2} \sup_{x \in \mathbb{R}^2} \langle x \rangle^s \mathbb{P}_x(\sigma(K) < t)^{1/2} \|f\|_2.$$

The known expression for the Brownian motion transition density yields that

$$t \mapsto \sup_{y \in K} \sup_{0 \leq \tau \leq t} (\mathbb{E}_y \langle X_\tau \rangle^{2s})^{1/2}$$

is $O(1)$ as $t \rightarrow 0+$ and $O(t^{s/2})$ as $t \rightarrow \infty$. The function

$$t \mapsto \sup_{x \in \mathbb{R}^2} \langle x \rangle^s \mathbb{P}_x(\sigma(K) < t)^{1/2}$$

has the same behaviour as can be seen using the "principle of not feeling the boundary". Thus the above integral is indeed finite. \square

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